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ON THE LENGTHS OF THE PIECES OF A STICK BROKEN AT RANDOM

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Abstract

Consider the pieces of a randomly broken stick. How long is the jth longest piece? How many breaks are necessary for getting all pieces less than a given length? These and related questions are studied in particular when the number of pieces is large. Using simple properties of the exponential distribution new proofs are given of old results and new results are obtained.

SPACINGS; UNIFORM DISTRIBUTION; EXPONENTIAL DISTRIBUTION; COVERING A CIRCLE; ORDER STATISTICS; LIMIT THEOREMS

1. Introduction

A stick of length 1 is broken at random into n pieces. To be more specific, let the stick be the unit interval and the breaking points be given by a random sample of size $n-1$ from the uniform distribution on this interval. These points divide the stick into n random intervals whose lengths we denote in order of magnitude by $S_{(1)} < S_{(2)} < \cdots < S_{(n)}$.

In Section 2 the exact distribution of $S_{(i)}$ is derived. Also moments of $S_{(i)}$ are obtained and some useful representations using independent exponential random variables are given.

For j fixed convergence of the distribution and the moments of the upper extreme value $S_{(n-j)}$, suitably normalized, is proved in Section 3. The same questions are studied in Section 4 for the lower extreme value $S_{(j)}$.

In Section 5 a necessary and sufficient condition for asymptotic normality is given, and convergence of moments is obtained for linear combinations of $S_{(1)}, \cdots, S_{(n)}$.

In the last section the number of points N_α in order to get length less than α for each interval is studied. A limit distribution for N_α is derived and convergence of moments is proved when $\alpha \to 0$. The random variable N_α can also

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be interpreted as the minimum number of arcs of length α placed at random on a circle of length 1 in order to get complete coverage.

Some words about notation. The interval lengths ordered along the line will be denoted by $S_1, \cdots, S_n$. Independent exponential random variables with mean 1 will always be denoted by $X_1, X_2, \cdots, X_n$ and then order statistics by $X_{(1)}, \cdots, X_{(n)}$.

The random variables $S_1, \cdots, S_n$ are usually called spacings on which a huge literature exists, see Pyke (1965), (1972), and Rao (1976) and the references therein. But the author has not found any systematic study of the order statistics. Further references will be given in connection with the different questions considered below.

2. Exact results

The following theorem and variants of it has been proved in different ways and contexts, see e.g. Whitworth (1897), Problem 667, Fisher (1929), (1940), Stevens (1939), Darling (1953), Flatto and Konheim (1962), Kendall and Moran (1963), p. 31, and Feller (1966), p. 28.

Theorem 2.1.

$$P(S_{(n-j)} \leqq x) = \sum_{\mu=0}^{j} \binom{n}{\mu} \sum_{\nu=0}^{n-\mu} (-1)^\nu \binom{n-\mu}{\nu} (1-(\mu+\nu)x)_+^{n-1}$$

where $a_+ = \max(a, 0)$.

Proof. Using indicator functions set $I_k = I(S_k > x)$. Evidently we have

$$P(S_{(n-j)} \leqq x) = P\left(\sum_{k=1}^{n} I_k \leqq j\right).$$

By symmetry the random vector $(I_1, \cdots, I_n)$ is exchangeable. Therefore

$$\begin{aligned} P\left(\sum_1^n I_k = \mu\right) &= \binom{n}{\mu} E\left(\prod_1^\mu I_k \prod_{\mu+1}^n (1-I_k)\right) \\ &= \binom{n}{\mu} \sum_{\nu=0}^{n-\mu} \binom{n-\mu}{\nu} (-1)^\nu E\left(\prod_1^{\mu+\nu} I_k\right). \end{aligned}$$

From this the assertion follows if we can show that

$$E\left(\prod_1^r I_k\right) = P(S_1 > x \cap \cdots \cap S_r > x) = (1-rx)_+^{n-1}.$$

This can be seen in the following way. Let $r = 2$, the general case is proved analogously. Consider

$$P(S_1 > x \cap S_2 > x) = P(S_1 > x)P(S_2 > x \mid S_1 > x).$$

The event $\{S_1 > x\}$ means that all $n-1$ breaking points are randomly distributed in the interval $(x, 1)$. Given that event the probability that the distance between the first and the second point exceeds x is by symmetry the same as the probability that the distance from the point x to the first point exceeds x. Thus

$$P(S_2 > x \mid S_1 > x) = P(\text{all points in } (2x, 1) \mid \text{all points in } (x, 1))$$

and therefore

$$P(S_1 > x \cap S_2 > x) = P(\text{all points in } (2x, 1)) = P(S_1 > 2x) = (1-2x)_+^{n-1}$$

which completes the proof.

Using simple properties of the Poisson process the following well-known results for the exponential distribution are easily proved, see e.g. Feller (1966), Sections I.6, III.3 and Pyke (1965), Sections 4.2–4.4.

For independent exponential random variables $X_1, \cdots, X_n$ with mean 1 and with

$$T_n = \sum_{k=1}^{n} X_k$$

we have

$$(S_1, \cdots, S_n) \sim (X_1/T_n, \cdots, X_n/T_n)$$

where $\sim$ means 'have the same distribution as'. Furthermore $(X_1/T_n, \cdots, X_n/T_n)$ and T_n are independent and T_n is gamma $(n, 1)$-distributed. Hence

$$(S_{(1)}, \cdots, S_{(n)}) \sim (X_{(1)}/T_n, \cdots, X_{(n)}/T_n).$$

It is also well known that order statistics from the exponential distribution have the representation given by

$$X_{(i)} \sim X_n/n + X_{n-1}/(n-1) + \cdots + X_{n-i+1}/(n-i+1).$$

Theorem 2.2.

$$E(S_{(i)}^r) = E(X_{(i)}^r)\Gamma(n)/\Gamma(n+r)$$

and

$$E(nS_{(i)}) = E(X_{(i)}) = \sum_{\nu=0}^{i-1} 1/(n-\nu).$$

Proof. As the random variables T_n and $X_{(i)}/T_n$ are independent and T_n is gamma$(n, 1)$ we have

$$\begin{aligned} E(X_{(i)}^r) &= E((X_{(i)}/T_n)^r T_n^r) = E((X_{(i)}/T_n)^r)E(T_n^r) \\ &= E(S_{(i)}^r)\Gamma(n+r)/\Gamma(n) \end{aligned}$$

proving the first assertion.

Taking $r = 1$ and using the representation for $X_{(i)}$ given above the second assertion follows.

We may remark that the method used above is very convenient for calculating arbitrary mixed moments of spacings, cf. Pyke (1965), Section 2.1, 4.4.

3. Asymptotic results for upper extreme values

The exact distribution for $S_{(n-j)}$ given by Theorem 2.1 is unsuitable for numerical calculations if n is large and x small. Therefore approximations are of interest. The asymptotic distribution of $S_{(n)}$ has been obtained by e.g. Lévy (1939), Darling (1953), and LeCam (1958). In LeCam (1958), p. 14 the asymptotic distribution for $S_{(n-j)}$ is implicitly given. We will give a simple derivation using a method based on the representation $S_{(n-j)} \sim X_{(n-j)}/T_n$, (cf. Holst (1980), Theorem 1) where a more complicated situation is considered. The notation $\xrightarrow{D}$ is used for convergence in distribution.

Theorem 3.1. For every fixed $j = 0, 1, 2, \cdots$

$$nS_{(n-j)} - \ln n \xrightarrow{D} Z_j, \qquad n \to \infty,$$

where

$$P(Z_j \leqq x) = \sum_{\nu=0}^{j} (e^{-x})^{\nu} e^{-e^{-x}}/\nu!.$$

Proof. From the representation $S_{(n-j)} \sim X_{(n-j)}/T_n$ we have

$$P(nS_{(n-j)} - \ln n \leqq x) = P(X_{(n-j)} - \ln n \leqq x + (x + \ln n)(\bar{T}_n - 1)),$$

where $\bar{T}_n = T_n/n = \Sigma_1^n X_k/n$. As $E(\bar{T}_n) = 1$ and $\mathrm{Var}(\bar{T}_n \ln n) = (\ln n)^2/n \to 0$ the random variable $nS_{(n-j)} - \ln n$ has the same asymptotic distribution as $X_{(n-j)} - \ln n$. Now

$$P(X_{(n-j)} \leqq x + \ln n) = P(Y_n \leqq j),$$

where by the independence of the X's

$$Y_n = \sum_{k=1}^{n} I(X_k > x + \ln n) \sim \text{binomial}(n, P(X_1 > x + \ln n)).$$

As

$$EY_n = n \cdot P(X_1 > x + \ln n) = e^{-x}$$

it follows that

$$Y_n \xrightarrow{D} \text{Poisson}(e^{-x}), \qquad n \to \infty.$$

Hence

$$P(X_{(n-j)} - \ln n \leqq x) \to \sum_{\nu=0}^{j} (e^{-x})^{\nu} e^{-e^{-x}}/\nu!.$$

As $nS_{(n-j)} - \ln n$ has the same asymptotic distribution, the theorem is proved.

Next convergence of moments is considered.

Theorem 3.2. For fixed $j, r = 0, 1, 2, \cdots$ and Z_j defined as in Theorem 3.1

$$E(nS_{(n-j)} - \ln n) \to EZ_j = \gamma - \sum_{i=1}^{j} 1/i,$$

where γ is Euler's constant, and

$$E((nS_{(n-j)} - E(nS_{(n-j)}))^r) \to E((Z_j - EZ_j)^r),$$

when $n \to \infty$.

Proof. The first assertion follows from Theorem 2.2 and the well-known limit

$$\sum_{1}^{n} 1/i - \ln n \to \gamma, \qquad n \to \infty,$$

cf. the discussion after Theorem 3.3 below.

By the results of Section 2 we have

$$\begin{aligned}
E((nS_{(n-j)} - E(nS_{(n-j)}))^r) &= E((X_{(n-j)}/\bar{T}_n - E(X_{(n-j)}))^r) \\
&= \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu E((X_{(n-j)}/\bar{T}_n)^\nu)(E(X_{(n-j)}))^{r-\nu} \\
&= \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu E(\bar{T}_n^\nu)^{-1} E(X_{(n-j)}^\nu)(E(X_{(n-j)}))^{r-\nu} \\
&= \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu E(X_{(n-j)}^\nu)(E(X_{(n-j)}))^{r-\nu} + O((\ln n)^r/n) \\
&= E((X_{(n-j)} - EX_{(n-j)})^r) + O((\ln n)^r/n) \\
&\to E((Z_j - EZ_j)^r), \qquad n \to \infty,
\end{aligned}$$

because by Theorem 3.3 below $E(\bar{T}_n^\nu) = 1 + O(1/n)$, $EX_{(n-j)}^\nu = O((\ln n)^\nu)$ and the central moment converges.

Theorem 3.3. For fixed $j, r = 0, 1, 2, \cdots$ and z with $\operatorname{Re} z > -1$, when $n \to \infty$,

$$E(\exp(-z(X_{(n-j)} - \ln n))) \to E(e^{-zZ_j}) = \Gamma(1+z) \prod_{\nu=1}^{j} (1 + z/\nu),$$

where $\Pi_{\nu=1}^{0}(1 + z/\nu) \equiv 1$, and

$$E((X_{(n-j)} - \ln n)^r) \to E(Z_j^r),$$

with

$$E(Z_j^r) = (-1)^r \left(\frac{d}{dz}\right)^r \left(\Gamma(1+z) \prod_{\nu=1}^{j} (1 + z/\nu)\right)\bigg|_{z=0}$$

Proof. Using the representation of $X_{(i)}$ we have

$$E(e^{-zX_{(n-j)}}) = E(\exp(-z(X_n/n + \cdots + X_{j+1}/(j+1))))$$
$$= \prod_{\nu=j+1}^{n} (1 + z/\nu)^{-1}$$
$$= E(e^{-zX_{(n)}}) \prod_{\nu=1}^{j} (1 + z/\nu).$$

By the distribution function transformation we have

$$e^{-X_{(n)}} \sim S_1^*$$

where S_1^* is the minimum in a sample of size n from a uniform distribution on $(0,1)$. Hence

$$E(e^{-zX_{(n)}}) = E(S_1^{*z}) = E(X_1^z)/E(T_{n+1}^z)$$

and therefore when $n \to \infty$

$$E(e^{-z(X_{(n)}-\ln n)}) = E(X_1^z) \Big/ E\left(\left(\sum_1^{n+1} X_k/n\right)^z\right) \to EX_1^z = \Gamma(1+z).$$

Thus the first assertion is proved. The second follows from the first, because convergence of the Laplace transform in an open interval surrounding the origin implies convergence of the moments. The third assertion follows trivially.

From the proof follows in $\operatorname{Re} z > -1$ the existence of the following limit

$$\lim_{n\to\infty} n^z n! \Big/ \prod_{\nu=1}^{n} (\nu + z) = \int_0^\infty t^z e^{-t} dt.$$

Hence we have given a probabilistic proof of the fact that the two definitions of the gamma function by Euler and Gauss are equivalent.

By Theorem 3.3 the cumulant generating function of $X_{(n)} - \ln n$ converges to $\ln \Gamma(1-t)$ for $t < 1$. Thus the following hold:

$$E(X_{(n)} - \ln n) = \sum_1^n 1/i - \ln n \to \gamma = \frac{d}{dt} \ln \Gamma(1-t)\Big|_{t=0} = -\int_0^\infty (\ln u) e^{-u} du$$

and

$$\operatorname{Var}(X_{(n)} - \ln n) = \sum_1^n 1/i^2 \to \pi^2/6 = \frac{d^2}{dt^2} \ln \Gamma(1-t)\Big|_{t=0}$$
$$= \int_0^\infty (\ln u)^2 e^{-u} du - \left(\int_0^\infty (\ln u) e^{-u} du\right)^2.$$

We also get for $\operatorname{Re} z > -1$ that

$$e^{-\gamma z}\left(\prod_{\nu=1}^{\infty} (1 + z/\nu) e^{-z/\nu}\right)^{-1} = \Gamma(1+z).$$

Hence with probability one $\gamma + \Sigma_{\nu=1}^{\infty}(X_\nu - 1)/\nu$ is a random variable with distribution function $\exp(-e^{-x})$ and Laplace transform $\Gamma(1+z)$.

4. Asymptotic results for lower extreme values

The asymptotics of the lower extreme values $S_{(j)}$ for fixed $j = 1, 2, 3, \cdots$ will be considered now. Lévy (1939), Darling (1953), LeCam (1958), and Flatto and Konheim (1962) have studied convergence of the distributions.

Theorem 4.1. When $n \to \infty$

$$n^2 S_{(j)} \xrightarrow{\mathrm{D}} Y_j$$

and

$$E((n^2 S_{(j)})^r) \to E(Y_j^r)$$

where $Y_j \sim$ gamma $(j, 1)$.

Proof. By the representation of $S_{(j)}$

$$P(n^2 S_{(j)} \leqq x) = P(nX_{(j)} \leqq x + x(\bar{T}_n - 1)).$$

As $\bar{T}_n \to 1$ in probability it is sufficient to consider the convergence of $nX_{(j)}$. But

$$nX_{(j)} \sim n(X_1/n + X_2/(n-1) + \cdots + X_j/(n-j+1)) \xrightarrow{\mathrm{D}} X_1 + X_2 + \cdots + X_j \sim Y_j$$

proving the first statement.

By Theorem 2.2 we have

$$E((n^2 S_{(j)})^r) = E((nX_{(j)})^r)/E(\bar{X}_n^r) \to E(Y_j^r)$$

because

$$\begin{aligned} E((nX_{(j)})^r) &= E((X_1 + X_2 n/(n-1) + \cdots + X_j n/(n-j+1))^r) \\ &\to E(X_1 + \cdots + X_j)^r = EY_j^r. \end{aligned}$$

5. Asymptotics for linear combinations

Linear combinations of order statistics of independent identically distributed random variables have been widely studied. In the special case of order statistics from the uniform distribution Hecker (1976) gave necessary and sufficient conditions for convergence to $N(0,1)$, the standard normal distribution. The corresponding results for linear combinations of ordered spacings will be obtained now.

Theorem 5.1.

$$\sum_1^n a_i (nS_{(i)} - E(nS_{(i)})) \xrightarrow{\mathrm{D}} N(0,1)$$

if and only if

$$\max_{1 \leq i \leq n} (b_i - \bar{b})^2 \to 0$$

and

$$\sum_1^n (b_i - \bar{b})^2 \to 1$$

where

$$b_i = \sum_{j=i}^n a_j / (n - i + 1), \qquad i = 1, 2, \cdots, n,$$

and

$$\bar{b} = \sum_1^n b_i / n.$$

Proof. Using the representation $S_{(i)} \sim X_{(i)} / T_n$ we find

$$\sum_1^n a_i S_{(i)} \sim \sum_1^n a_i X_{(i)} / T_n \sim T_n^{-1} \cdot \sum_{i=1}^n X_i \sum_{j=i}^n a_j / (n - i + 1) \sim \sum_1^n b_i S_i .$$

Now $E(nS_i) = 1$ and $\Sigma_1^n S_i = 1$ and therefore

$$\sum_1^n a_i (nS_{(i)} - E(nS_{(i)})) \sim \sum_1^n b_i (nS_i - 1) = \sum_1^n (b_i - \bar{b}) nS_i \sim \sum_1^n (b_i - \bar{b}) X_i / \bar{T}_n .$$

Because $\bar{T}_n \to 1$ in probability, we have

$$\sum_1^n (b_i - \bar{b}) X_i / \bar{T}_n \xrightarrow{\mathrm{D}} N(0, 1)$$

if and only if

$$\sum_1^n (b_i - \bar{b})(X_i - 1) \xrightarrow{\mathrm{D}} N(0, 1).$$

It is easily seen that the Lindeberg condition is equivalent to

$$\max_{1 \leq i \leq n} (b_i - \bar{b})^2 \Big/ \sum_1^n (b_i - \bar{b})^2 \to 0.$$

Therefore, by the Lindeberg–Feller theorem, see e.g. Feller (1966), Section XV.6,

$$\sum_1^n (b_i - \bar{b})(X_i - 1) \xrightarrow{\mathrm{D}} N(0, 1)$$

if and only if

$$\max_{1 \leq i \leq n} | b_i - \bar{b} | \to 0, \qquad \sum_1^n (b_i - \bar{b})^2 \to 1,$$

proving the theorem.

Theorem 5.2. Let $b_1, b_2, \cdots, b_n$ be defined as in Theorem 5.1 and satisfy

$$\max |b_i - \bar{b}| \to 0, \qquad \sum_1^n (b_i - \bar{b})^2 \to 1.$$

Then for $r = 1, 2, \cdots$

$$E\left(\left(\sum_1^n a_i (nS_{(i)} - E(nS_{(i)}))\right)^r\right) \to E(Y^r)$$

where $Y \sim N(0, 1)$.

Proof. We have as in Theorem 5.1 that

$$E\left(\left(\sum_1^n a_i (nS_{(i)} - E(nS_{(i)}))\right)^r\right) = E\left(\left(\sum_1^n (b_i - \bar{b})X_i / \bar{T}_n\right)^r\right)$$

$$= E\left(\left(\sum_1^n (b_i - \bar{b})X_i\right)^r\right) \Big/ E(\bar{T}_n^r)$$

by the independence between $(X_1/T_n, \cdots, X_n/T_n)$ and T_n. As $E(\bar{T}_n^r) \to 1$ it is sufficient to show that

$$E\left(\left(\sum_1^n (b_i - \bar{b})X_i\right)^r\right) \to E(Y^r).$$

By the conditions on the b's it follows that for z in an open interval surrounding the origin

$$E\left(\exp\left(-z\sum_1^n (b_i - \bar{b})X_i\right)\right) = \left(\prod_1^n (1 + z(b_i - \bar{b}))\right)^{-1}$$

$$= \exp\left(-\sum_1^n \ln(1 + z(b_i - \bar{b}))\right)$$

$$= \exp(z^2/2 + o(1)).$$

Hence the Laplace transform converges for small $|z|$ to that of the $N(0, 1)$, implying the convergence of the moments given above. Thus the theorem is proved.

Note that asymptotic normality and convergence of moments of central order statistics $nS_{(j_n)}$, j_n and $n - j_n \to +\infty$, are consequences of the above results.

It is seen from the proofs that the random variable $\Sigma_1^n a_i (nS_{(i)} - E(nS_{(i)}))$ has the same asymptotic behaviour as $\Sigma_1^n a_i (X_{(i)} - E(X_{(i)}))$. We can say that the dependence among the S's coming from $\Sigma_1^n S_k = 1$ does not matter. In general, this dependence is important when considering asymptotic results for spacings. See e.g. LeCam (1958) where the asymptotic distributions are found for random variables of the type

$$\sum_1^n h_n(nS_k) = \sum_1^n h_n(nS_{(i)}),$$

where $h_n(\cdot)$ is a function depending on n.

Darling (1953) has also considered such random variables. This is also illustrated by Rao and Sethuraman (1975), Theorem 3.1, showing that the empirical distribution function of the spacings suitably normalized converges weakly to a Gaussian process with covariance kernel $e^{-y}(1-e^{-x}-xye^{-x})$, $0 \leqq x \leqq y$. The covariance kernel for the corresponding process based on independent exponential random variables is $e^{-y}(1-e^{-x})$.

6. The waiting time N_α

Let N_α be the minimum number of pieces until they all have a length at most α. Evidently

$$P(N_\alpha \leqq n) = P(S_{(n)} \leqq \alpha).$$

By Theorem 3.1 it follows that

$$P(N_\alpha \leqq n) = P(nS_{(n)} - \ln n \leqq n\alpha - \ln n) \to \exp(-e^{-x}),$$

if $n \to \infty$ and $\alpha \to 0$ such that $n\alpha - \ln n \to x$ or equivalently if

$$n\alpha - \ln(1/\alpha) - \ln\ln(1/\alpha) \to x.$$

Hence we have proved the following result.

Theorem 6.1. When $\alpha \to 0$,

$$P(\alpha N_\alpha - \ln(1/\alpha) - \ln\ln(1/\alpha) \leqq x) \to \exp(-e^{-x}).$$

The problem of finding the limit distribution of N_α was posed by Shepp (1972) in a paper on covering a circumference with arcs placed at random. In that context N_α can be interpreted as the minimum number of arcs, each of length α, for complete coverage of a circumference of length 1. In Flatto (1973), Edens (1975), Kaplan (1977), and Holst (1980) the limit distribution of the number of arcs to cover m times is obtained by different methods. The method used above follows Holst (1980).

Flatto and Konheim (1962) showed that $EN_\alpha \sim (1/\alpha)\ln(1/\alpha)$, the leading term of the asymptotic distribution. This was improved by Steutel (1967) proving

$$EN_\alpha = (1/\alpha)(\ln(1/\alpha) + \ln\ln(1/\alpha) + \gamma + o(1))$$

and by Edens (1975) showing that all the normalized moments of N_α converge to the corresponding of the limiting distribution $\exp(-e^{-x})$. Using Theorem 3.2 we will prove this in another way.

Theorem 6.2. Let Z have the distribution function $\exp(-e^{-x})$. Then

$$E((\alpha N_\alpha - \ln(1/\alpha) - \ln\ln(1/\alpha))^r) \to EZ^r = (-1)^r \Gamma^{(r)}(1).$$

Proof. Let $A > 0$ be fixed and $x > A$.

$$\begin{aligned} &P(\alpha N_\alpha - \ln 1/\alpha - \ln\ln 1/\alpha > x) \\ &\quad = P(S_{([(\ln 1/\alpha + \ln\ln 1/\alpha + x)/\alpha + 1])} > \alpha) \\ &\quad = P([\cdots]S_{([\cdots])} - \ln[\cdots] > \alpha[\cdots] - \ln[\cdots]) \\ &\quad \leqq P([\cdots]S_{([\cdots])} - \ln[\cdots] > x - \ln A(\alpha, x)), \end{aligned}$$

where for $0 < \alpha < \alpha_0$

$$A(\alpha, x) = \ln(1 + (\alpha + \ln\ln 1/\alpha)/\ln 1/\alpha + x/\ln 1/\alpha) < x/2.$$

Thus, by Theorem 3.2 we have for every $r = 1, 2, \cdots$ that

$$\begin{aligned} &P(\alpha N_\alpha - \ln 1/\alpha - \ln\ln 1/\alpha > x) \\ &\quad \leqq P([\cdots]S_{([\cdots])} - \ln[\cdots] > x/2) \\ &\quad \leqq E(|[\cdots]S_{[\cdots]} - \ln[\cdots]|^r) 2^r/x^r \leqq K_r/x^r. \end{aligned}$$

Therefore we have uniformly for $\alpha < \alpha_0$ the bound on the right tail

$$P(\alpha N_\alpha - \ln 1/\alpha - \ln\ln 1/\alpha > x) \leqq K_r/x^r.$$

In an analogous way one gets a bound on the left tail. Then for every $r = 1, 2, \cdots$

$$P(|\alpha N_\alpha - \ln 1/\alpha - \ln\ln 1/\alpha| > x) \leqq K_r/x^r$$

uniformly in $0 < \alpha < \alpha_0$. This implies the uniform integrability of $(\alpha N_\alpha - \ln 1/\alpha - \ln\ln 1/\alpha)^r$. Hence the assertion follows from Theorem 6.1.

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