Max PlanckInstitute for Gravitational Physics
IMPRS Lecture Series

Making sense of data: introduction to statistics for gravitational wave astronomy

Problem Sheet 1: Frequentist Statistics

Questions marked with a * are a selection that will give experience of all aspects of the course. For IMPRS students taking this course, these should be completed and handed in to be marked.

1. Show that the t_n distribution with pdf

$$p(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}.$$

is properly normalised, i.e., the integral of the pdf is 1.

2. * For the Beta(a, b) distribution, find the mean, mode, variance, skewness and excess kurtosis.

3. Derive the moment generating function for the exponential distribution, $\mathcal{E}(\lambda)$ and the Gamma(n, λ) distribution. Hence deduce that the distribution of the sum of n IID $\mathcal{E}(\lambda)$ random variables is Gamma(n, λ).

4. * Suppose $X \sim N(0, 1)$ with pdf

$$p(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}$$

and $Y \sim \chi^2_n$ with pdf

$$p(y) = \frac{1}{2^{\frac{n}{2}}\Gamma(n/2)} y^{\frac{n}{2}-1} \mathrm{e}^{-\frac{y}{2}}.$$

Show that the distribution of $T = X/\sqrt{Y/n}$ has pdf

$$p(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}.$$

This is the Student t-distribution with n degrees of freedom.

5. *Gravitational Wave Birthday Problem(s):

(a) How many gravitational wave sources would we have to observe before it is more likely than not we will have two events on the same date (i.e., day and month)?

(b) Suppose we have observed n GW events in a particular category, say binary black hole mergers and then observe an event in a new category. What is the probability that the new event is on the same date as one of the previously observed events (consider both the case that we know all the events in the category are on different days, and the case where this is not specified)?

(c) Given a rate of gravitational wave events of one per week, how many events would we have to observe before having a greater than 50% chance that two events were observed within 24 hours? [Hint: to answer the first question consider the distribution of the minimum difference between successive events and compute the probability that this is less than 1 day.]

(d) (OPTIONAL) Given a rate of gravitational wave events of one per week, how long would we expect to wait before having a greater than 50% chance of observing two events in 24 hours? This latter question is considerably more difficult to answer than the previous one, but gives a very similar answer.

6. **Gravitational wave physicist birthday cake problem:** It is traditional at the Alfred Embleton Institute for gravitational wave physics that when one member of the institute has a birthday, they bring cake to share with the other members of the group. One student, Andrew Antony, is very fond of cake and would like to eat it at least once every two weeks.

   (a) Given that the institute has n members, compute the probability distribution of the maximum separation between birthdays. How large must n be such that the probability that the maximum separation is less than two weeks is greater than 50%?

   (b) The director of the institute, Alice Bunton, is concerned that the cakes are bad for the health of the researchers in her institute, and therefore wants to make sure these celebrations do not occur too often. Find the distribution of the minimum separation between birthdays. What is the maximum n should be to ensure the probability that the minimum separation is greater than 2 weeks is at least 50%?

   [*Note: all similarities to real institutes and researchers are purely coincidental.*]

7. A life test is conducted by installing n items of equipment at time 0 and recording at times $h, 2h, \ldots, mh$ the numbers n_r of items failing in the intervals $(r-1)h$ to rh $(r = 1, 2, \ldots, m)$, m and h being a fixed integer and a fixed time interval respectively. The time to failure is modelled as an Exponential (λ) distribution, and the lifetimes of different items are assumed independent.

   (a) Find the likelihood function for λ.

   (b) Hence determine sufficient statistics for λ.

8. Let $X_1, X_2, \ldots, X_n$ be independent r.v.s where X_i has p.d.f. $\theta_i e^{-\theta_i x}$, $x > 0$ where $\theta_i = (\alpha + i\beta)$ and α, β are unknown parameters.

   Find sufficient statistics for (α, β).

9. * Independent Bernoulli r.v.s. $X_1, X_2, \ldots, X_n$ are such that the probability of X_i taking the value 1 depends on an explanatory variable z, which takes corresponding values $z_1, z_2, \ldots, z_n$.

   Show that for the model

$$\rho_j = \log\left\{\frac{\Pr(X_j = 1)}{\Pr(X_j = 0)}\right\} = \alpha + \beta z_j,$$

   the minimal sufficient statistic for (α, β) is $\left(\sum_{j=1}^n X_j,\ \sum_{j=1}^n z_j X_j\right)$; the quantity ρ_j is called the logistic transformation.

10. * Let $X_1, X_2, \ldots, X_n$ be a random sample from $U[0, \theta]$.

(a) Find the p.d.f. of $X_{(n)}$, the largest of the X_is.

(b) Show that $2\bar{X}$ (where $\bar{X}$ is the sample mean) and $(n+1)X_{(n)}/n$ are both unbiased consistent estimators of θ, and compare their variances.

11. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with p.d.f. $p(x|\,\lambda) = \lambda \mathrm{e}^{-\lambda x} \quad x > 0, \quad \lambda > 0.$

Find the maximum likelihood estimator, its mean and variance and the Cramer-Rao bound on the variance of unbiased estimators of λ. Hence show that the maximum likelihood estimator for λ is biased, consistent and asymptotically efficient.

12. * Suppose that $x_1, \ldots, x_n$ form a random sample from a distribution with probability density function

$$f(x|\sigma^2) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad (x > 0).$$

Obtain the Cramér-Rao lower bound when the parameter of interest is $\theta = \sigma^2$.

Determine whether the bound is attainable, and if it is attainable give the estimator which attains the bound.

13. Let $X_1, X_2, \ldots, X_n$ denote n independent, identically distributed random variables with a Bernoulli density $p(x|p) = p^x(1-p)^{1-x}$ for $x = 0, 1$. Show that X_1 is an unbiased estimator for p and compute its variance. Show that $S = \sum X_i$ is a sufficient statistic. Use the Rao-Blackwell theorem to obtain an estimator of lower variance and compute its variance.

14. **Linear modelling:** Consider observations

$$y_i = \beta^T \mathbf{x}_i + \epsilon_i$$

where $\epsilon_i \sim N(0, \sigma^2)$, β is a vector of parameters and $\mathbf{x}_i$ is a vector of k covariates for each observation y_i.

(a) Show that the maximum likelihood estimate for β is

$$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

where $\mathbf{X}$ is the *design matrix*, defined by $X_{ij} = (\mathbf{x}_i)_j$.

(b) Show that the distribution of $\hat{\beta}$ is

$$N\left(\beta, \sigma^2\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\right).$$

(c) Show that the quantity

$$\hat{\sigma}^2 = \frac{\mathbf{y}^T\mathbf{y} - \hat{\beta}^T\mathbf{X}^T\mathbf{y}}{n-k}$$

is an unbiased estimator of the variance σ^2. In fact it is relatively straightforward to show that this quantity is independent of $\hat{\beta}$ and follows a chi-squared distribution.

(d) For a fixed constant vector $\mathbf{c}$, show that

$$\frac{\mathbf{c}^T\hat{\beta} - \mathbf{c}^T\beta}{\hat{\sigma}\sqrt{\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{c}}}$$

follows a t-distribution and hence deduce a 95% confidence interval for $\mathbf{c}^T\beta$.

15. Let $X_1, X_2, \ldots, X_n$ denote n independent, identically distributed random variables having a Poisson distribution with mean λ.

(a) Derive the form of the most powerful test, of size α, of the simple null hypothesis $H_0 \colon \lambda = \lambda_0$ against the simple alternative hypothesis $H_1 \colon \lambda = \lambda_1$ $(\lambda_1 > \lambda_0)$.

Deduce the form of the uniformly most powerful (UMP) test of the simple hypothesis $H_0 : \lambda = \lambda_0$ against the composite alternative hypothesis $H_1 : \lambda > \lambda_0$.

(b) Determine the moment generating function of X_i, and hence show that $\sum X_i$ has a Poisson distribution with parameter $n\lambda$.

Explain how the distribution of $\sum X_i$ may be used to determine a critical region for the test in (a), and obtain the critical value for a test with a nominal level of 5% when $n = 10$ and $\lambda_0 = 1$. Compare this critical value with an approximate critical value obtained by using a normal approximation to the distribution of $\sum X_i$.

(c) Calculate the power of the test in (b) when $\lambda = 2$.

(d) Suppose now that we require a test of $H_0 \colon \lambda = \lambda_0$ against the alternative $H_1 \colon \lambda \neq \lambda_0$. Determine whether a uniformly most powerful test exists. Calculate (approximate) critical values of a two-sided 5% level test obtained by using a normal approximation to the distribution of $\sum X_i$ when $n = 10$ and $\lambda_0 = 1$

16. * Let $x_1, \ldots, x_n$ denote a random sample from a distribution with probability density function

$$p(x \mid \theta) = \frac{x}{\theta} \exp\left(-\frac{x^2}{2\theta}\right) \qquad (x > 0)$$

where θ is a positive constant.

(a) Obtain a minimal sufficient statistic for θ based on $x_1, \ldots, x_n$, and explain why it is minimal sufficient.

(b) Show that the most powerful test of size α of

$$\begin{aligned} H_0 &: \theta = \theta_0, \\ \text{against } H_1 &: \theta = \theta_1 \qquad (\theta_1 > \theta_0), \end{aligned}$$

involves a minimal sufficient statistic.

Deduce the form of the uniformly most powerful test of $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_1' : \theta > \theta_0$.

(c) Let $Y_i = X_i^2/\theta$, $i = 1, \ldots, n$, where X_i is defined as above. Show that Y_i has an exponential distribution with mean 2, i.e. a χ_2^2 distribution.

Deduce the critical value of the uniformly most powerful test of $H_0 : \theta = 1$ against $H_1' : \theta > 1$ in (b) when there are five observations and the size of the test is 5%. Find the power of the test as a function of θ.

17. * Let $x_1, \ldots, x_n$ be observations of independent random variables $X_1, \ldots, X_n$ from the distribution with the probability density function

$$p(x_i \mid \theta) = \frac{(z_i\theta)^a}{\Gamma(a)} x_i^{a-1} e^{-\theta z_i x_i}, \quad x_i > 0,$$

with known covariates $z_i > 0$ and known $a > 0$, that is, $X_i \sim \Gamma(a, z_i\theta)$.

(a) Derive the form of the most powerful test of size α, of the simple null hypothesis $H_0 : \theta = 1$ against the simple alternative hypothesis $H_1 : \theta = \theta_1 \quad (\theta_1 > 1)$.

(b) Deduce the form of the uniformly most powerful (UMP) test of the simple hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta > 1$.

(c) Does there exist a UMP test of the simple hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta \neq 1$?

(d) For observed data with $a = 2$, $n = 311$ and $\sum_i z_i y_i = 571$, test the hypothesis that $\theta = 1$ against $\theta > 1$.

[*Hint: use the Central Limit Theorem to find an approximate distribution of the test statistic.*]

(e) Find the power of the test $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 3$ as a function of n for $a = 2$ and $\alpha = 0.05$. Find the smallest n such that the power of the test is greater than 0.9.

[*Hint: use the Central Limit Theorem to find an approximate distribution of the test statistic.*]

(f) Determine the best critical regions of size α, of the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1 \quad (\theta_1 > \theta_0)$. Use these critical regions to construct a one-sided 90% confidence interval for θ for the data given in (d).