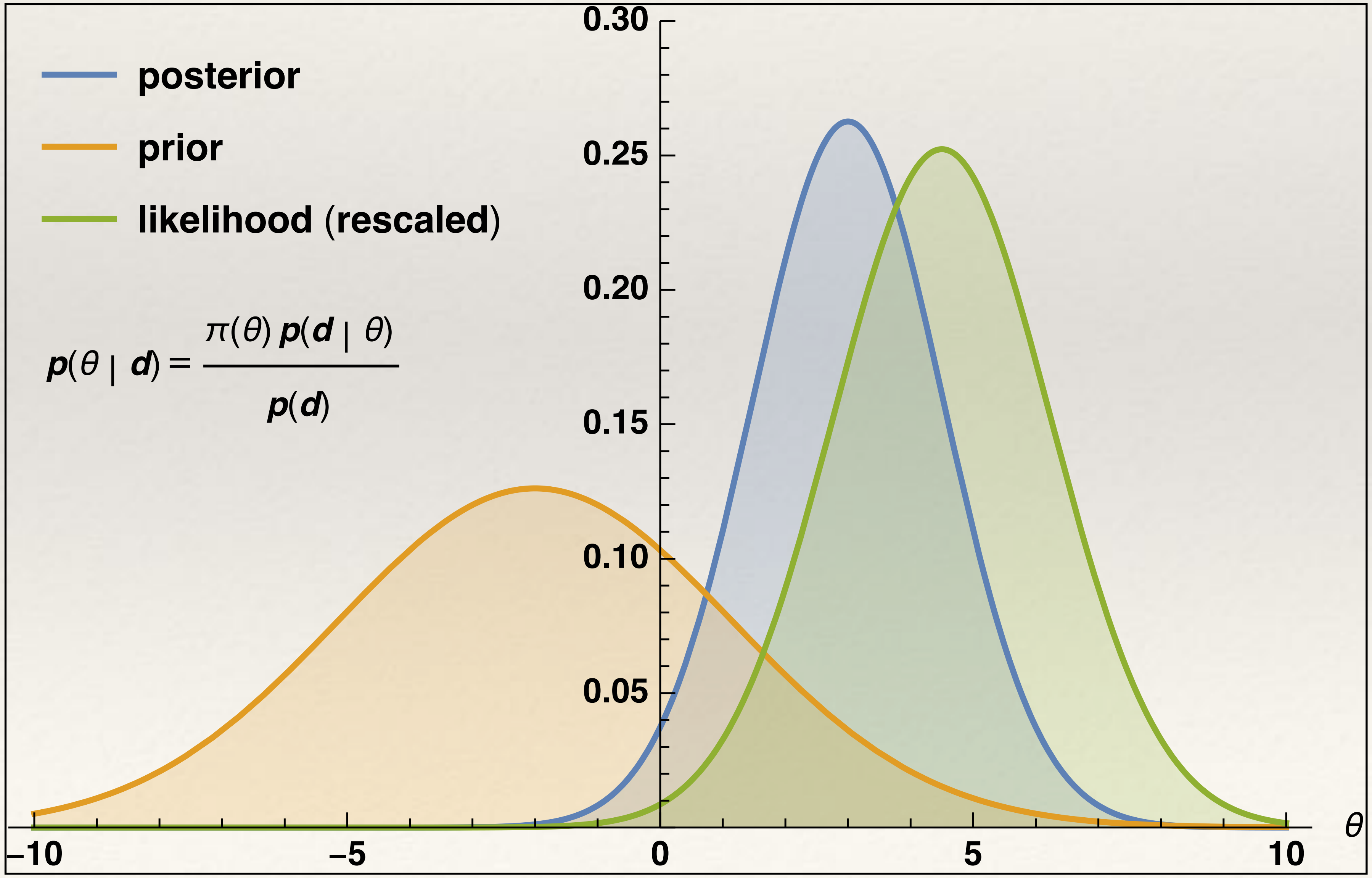
Making sense of data: introduction to statistics for gravitational wave astronomy

Lecture 1: introduction to random variables

AEI IMPRS Lecture Course

Jonathan Gair jgair@aei.mpg.de



Outline of course

- Lectures will take place at 11:30am Wednesday and Friday in the weeks beginning Nov 18th, 25th and Dec 2nd and 9th 2019 and the weeks beginning Jan 13th, 20th, 27th and Feb 3rd 2020.
- Lectures will all take place in seminar room 0.01 at the AEI and will be broadcast via Zoom
  - https://mpi-aei.zoom.us/j/867860487
- Lecture recordings and other material will be made available on the course website
  - https://imprs-gw-lectures.aei.mpg.de/potsdam-2019/

Outline of course

- **Section 1 (weeks 1 and 2): Frequentist statistics**
    - Random variables: definition, properties, some useful probability distributions, central limit theorem.
    - Statistics: definition, estimators, likelihood, desirable properties of estimators, Cramer-Rao bound.
    - Hypothesis testing: definition, Neyman-Pearson lemma, power and size of tests, type I and type II errors, ROC curves, confidence regions, uniformly-most-powerful tests.

Outline of course

- **Section 2 (weeks 3 and 4): Bayesian statistics**
  - Bayes' theorem, conjugate priors, Jeffrey's prior.
  - Bayesian hypothesis testing, hierarchical models, posterior predictive checks.
  - Sampling methods for Bayesian inference

Outline of course

- **Section 3 (weeks 5 and 6): Statistics in gravitational wave astronomy**
  - Stochastic processes, optimal filtering, signal-to-noise ratio, sensitivity curves.
  - Frequentist statistics in GW astronomy: false alarm rates, Fisher Matrix, PSD estimation.
  - Bayesian statistics in GW astronomy: parameter estimation, population inference, model selection.

Outline of course

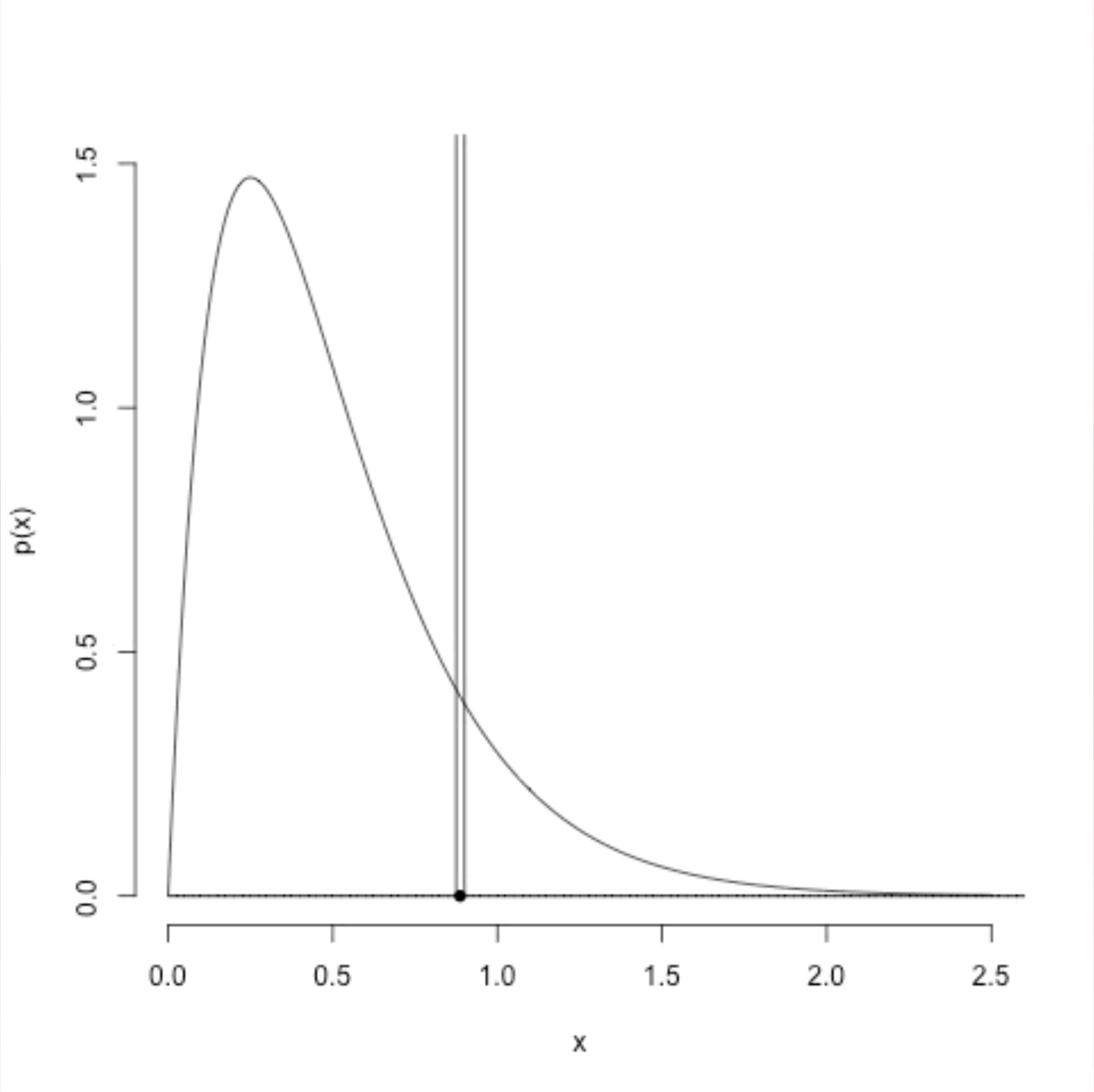
- **Section 4 (weeks 7 and 8): Advanced topics in statistics**
  - Time series analysis: auto-regressive processes, moving average processes, ARMA models.
  - Nonparametric regression: kernel density estimation, smoothing splines, wavelets.
  - Gaussian processes, Dirichlet processes.

Outline of course

- Every fourth lecture, which comes at the end of a block, will be a computational practical class, illustrating how to compute some of the quantities introduced in lectures in practice. These practicals will use python.
- One problem set will be provided for each block of lectures. Solutions will be made available later.

Random variables

- *Random variables* are quantities that are not fixed, but can take new values each time they are observed (a *realisation*).
- Over many realisations the distribution of the random variable is described by a *probability distribution*.
- Random variables can be *discrete* (taken values in a countable set) or *continuous* (taking real values in some interval).



Discrete random variables

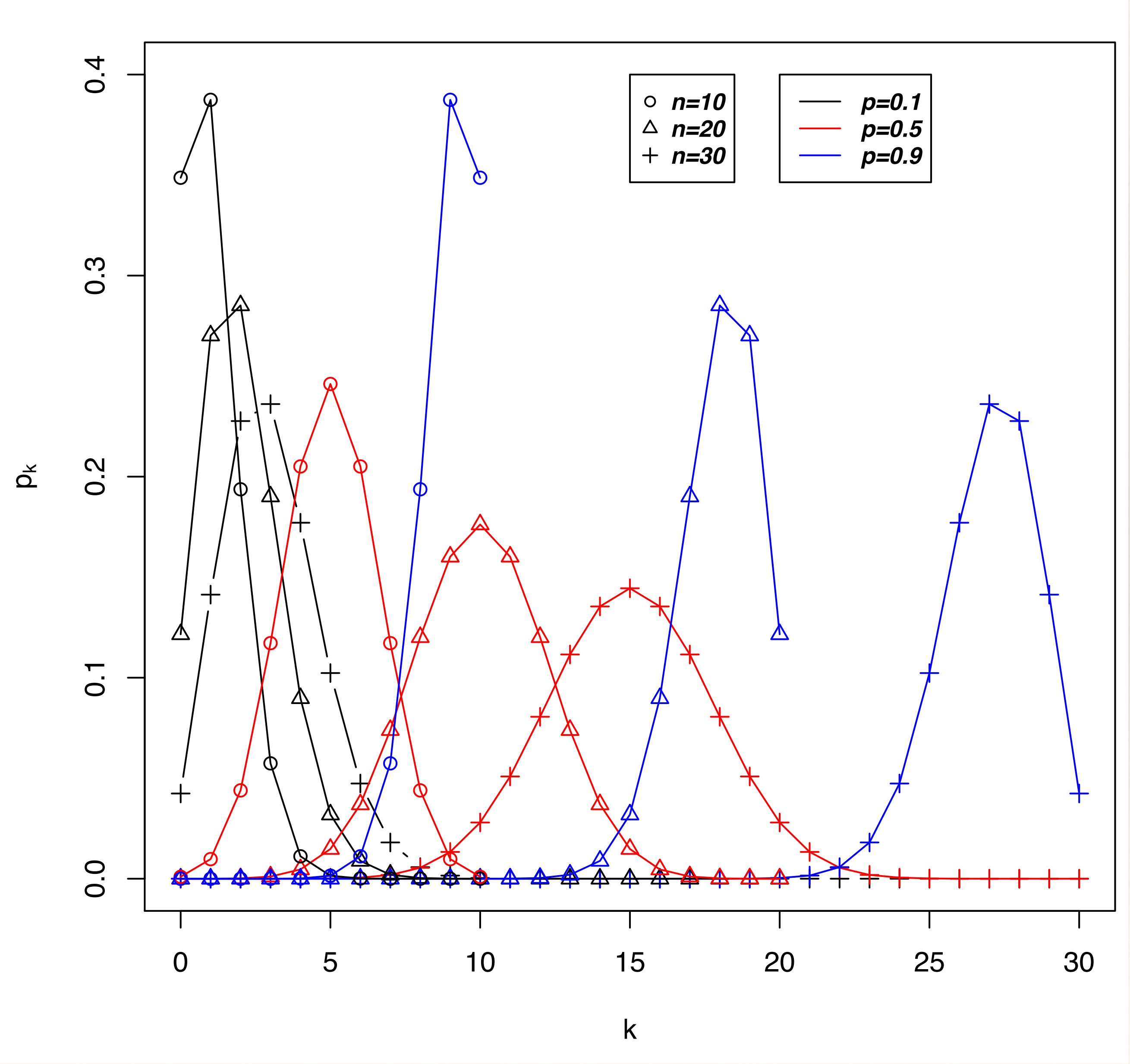
- Discrete random variables are characterised by a *probability mass function*, i.e., a set $\{p_i\}$ satisfying

$$0 \leq p_i \leq 1 \qquad \sum p_i = 1$$

- For example, **Binomial distribution**

$$P(X=k) = p_k = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{1, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$

- Related distributions: *Bernoulli distribution, negative Binomial, geometric distribution.*

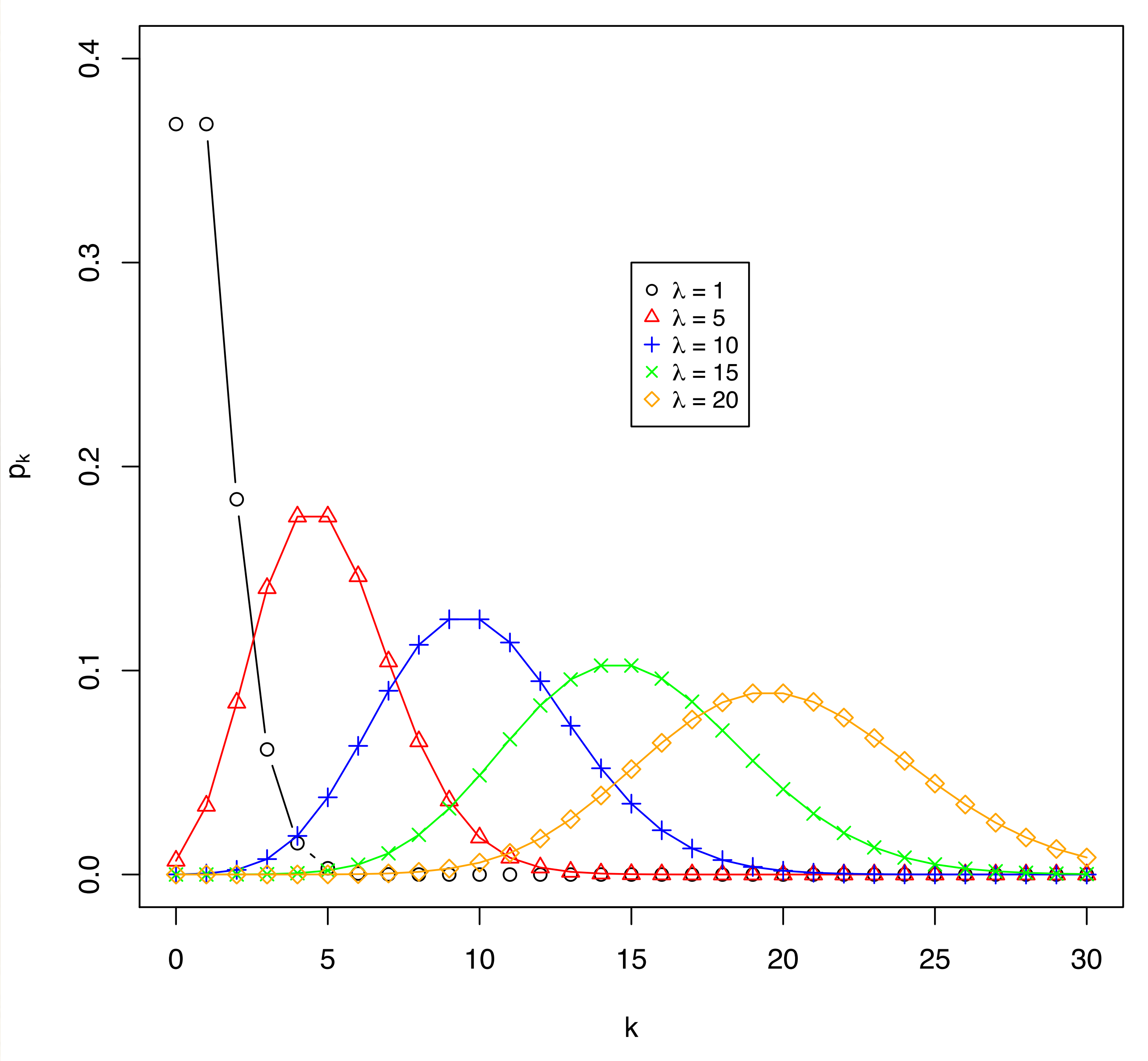


Discrete RVs: Poisson distribution

- **Poisson distribution** is defined for non-negative *k* by

$$P(X = k) = p_k = \begin{cases} \lambda^k \mathrm{e}^{-\lambda}/k! \\ 0 \end{cases}$$

- Arises as the distribution of the number of counts of a process occurring in a certain period of time.



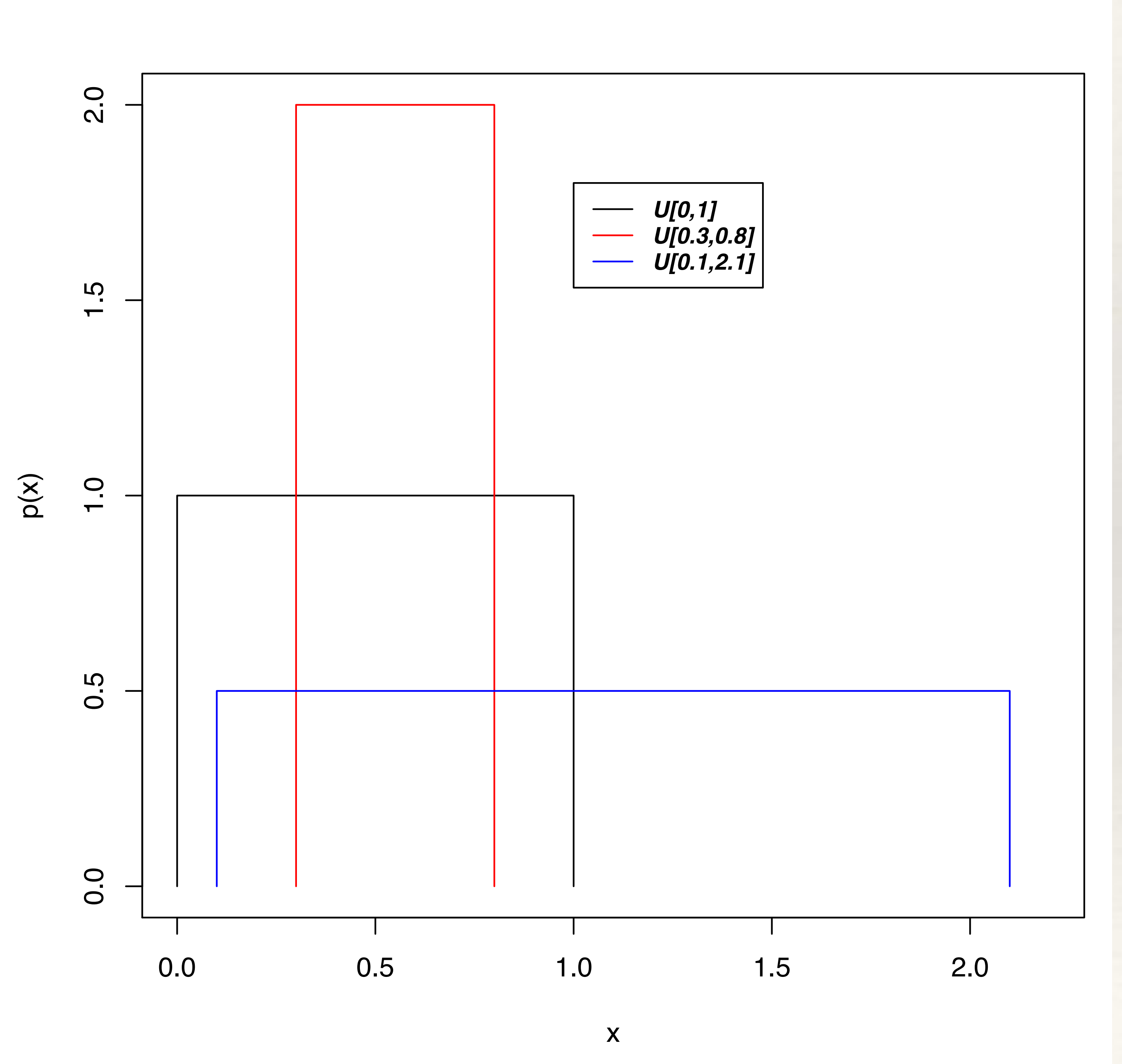
Continuous random variables

- Continuous random variables are characterised by a *probability density function*, satisfying

$$0 \leq p(x) \qquad \int_{x \in \mathcal{X}} p(x) \mathrm{d}x = 1$$

- For example, **Uniform distribution**

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$



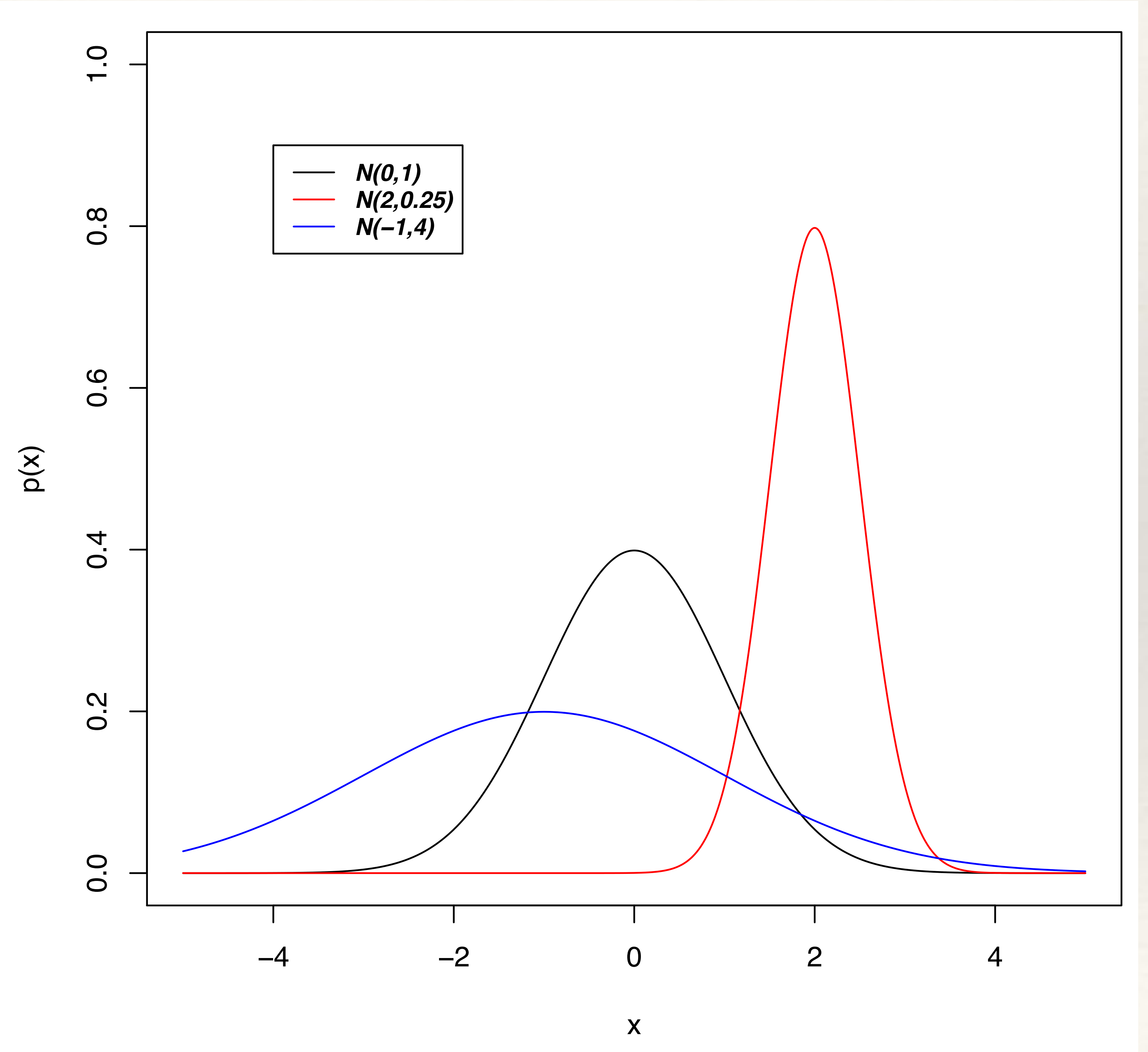
Continuous RVs: Normal distribution

- **Normal distribution** is characterised by mean μ and variance σ^2

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Arises as a limiting distribution and as the distribution of noise in gravitational wave detectors. Commonly used as the default distribution in parametric statistics and as a prior in Bayesian analysis.

- Normal distribution with zero mean and unit variance is the *standard Normal distribution*.

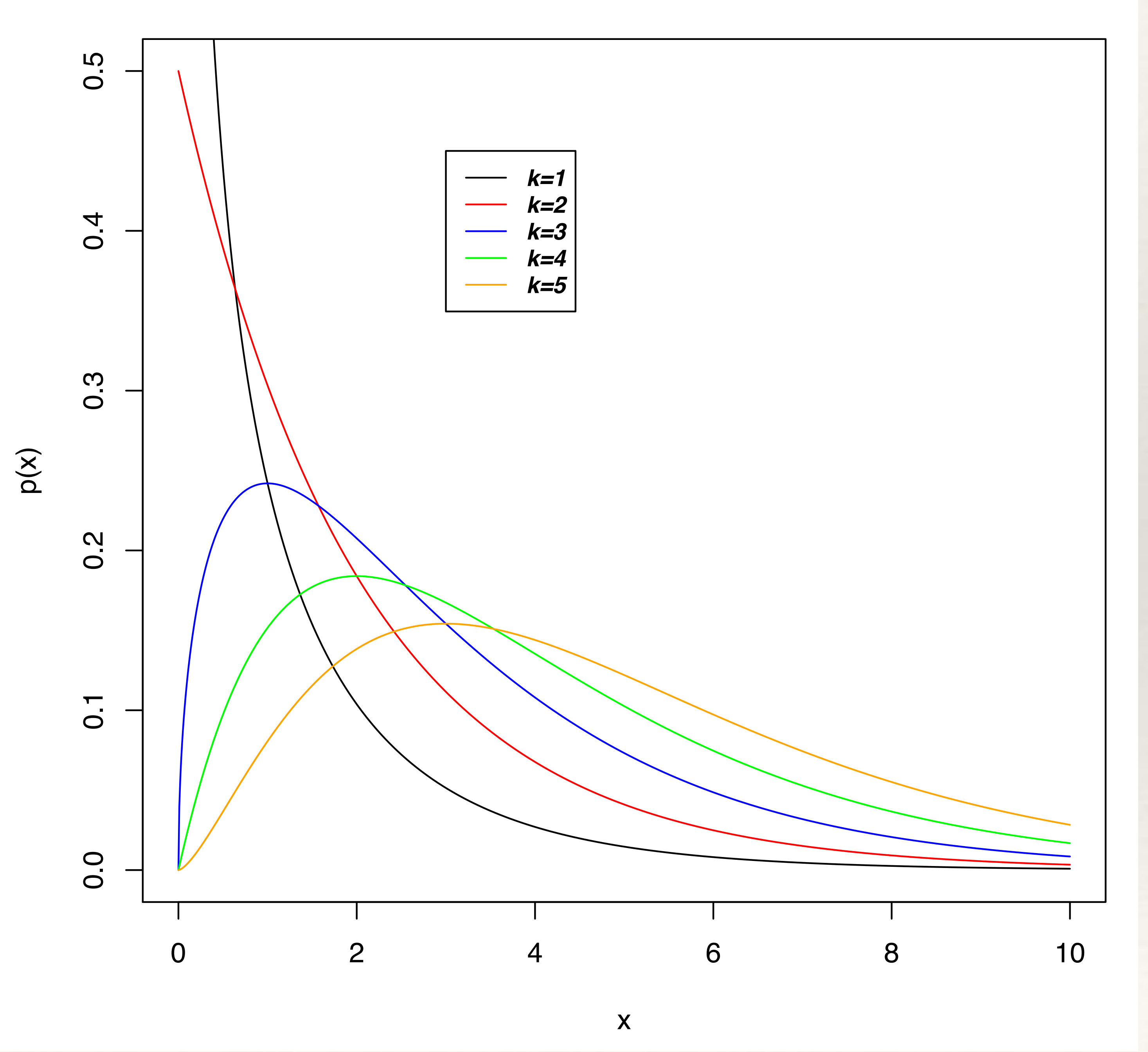


Continuous RVs: chi-squared distribution

- **Chi-squared distribution** depends on a *degrees of freedom* parameter k > *0*

$$p(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{\frac{k}{2}-1} \mathrm{e}^{-\frac{x}{2}}$$

- It is the distribution of the sum of squares of *k* standard normal random variables.
- There is also a *non-central chi-square distribution* which has also a *non-centrality parameter*.

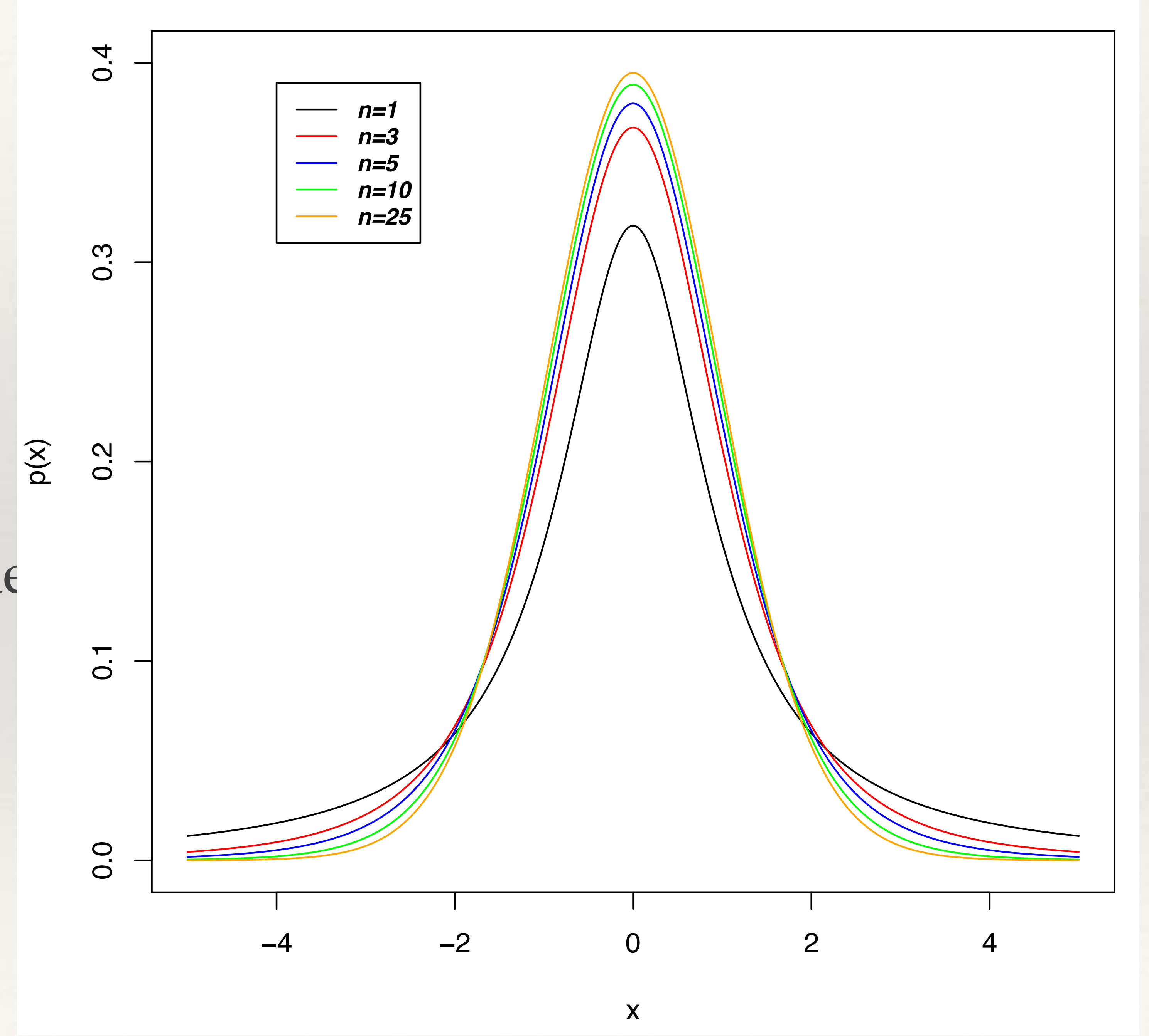


Continuous RVs: Student's t-distribution

- **Student's t-distribution** ***also*** depends on a degrees of freedom parameter *n*

$$p(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

- It arises in hypothesis testing as the ratio of a standard Normal distribution to a chi-squared distribution. It is used as a *heavy-tailed* distribution in inference.

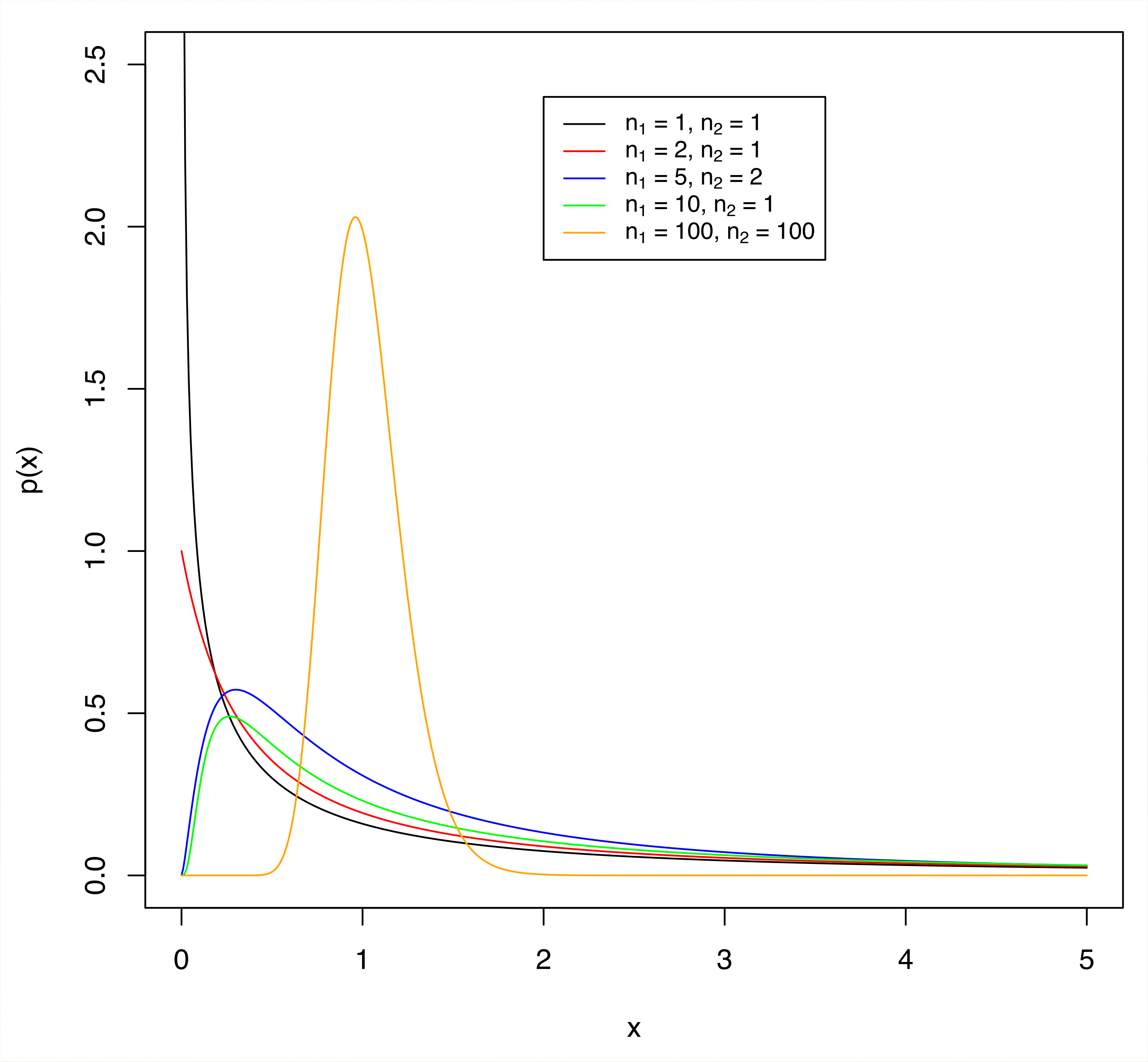


Continuous RVs: F-distribution

- The **F-distribution** depends on two degrees of freedom parameters, n_1 and n_2

$$p(x) = \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1} \left(1 + \frac{n_1}{n_2}x\right)^{-\frac{n_1+n_2}{2}}$$

- This arises as the ratio of two chi-square distributions and is the basis for *analysis of variance*.

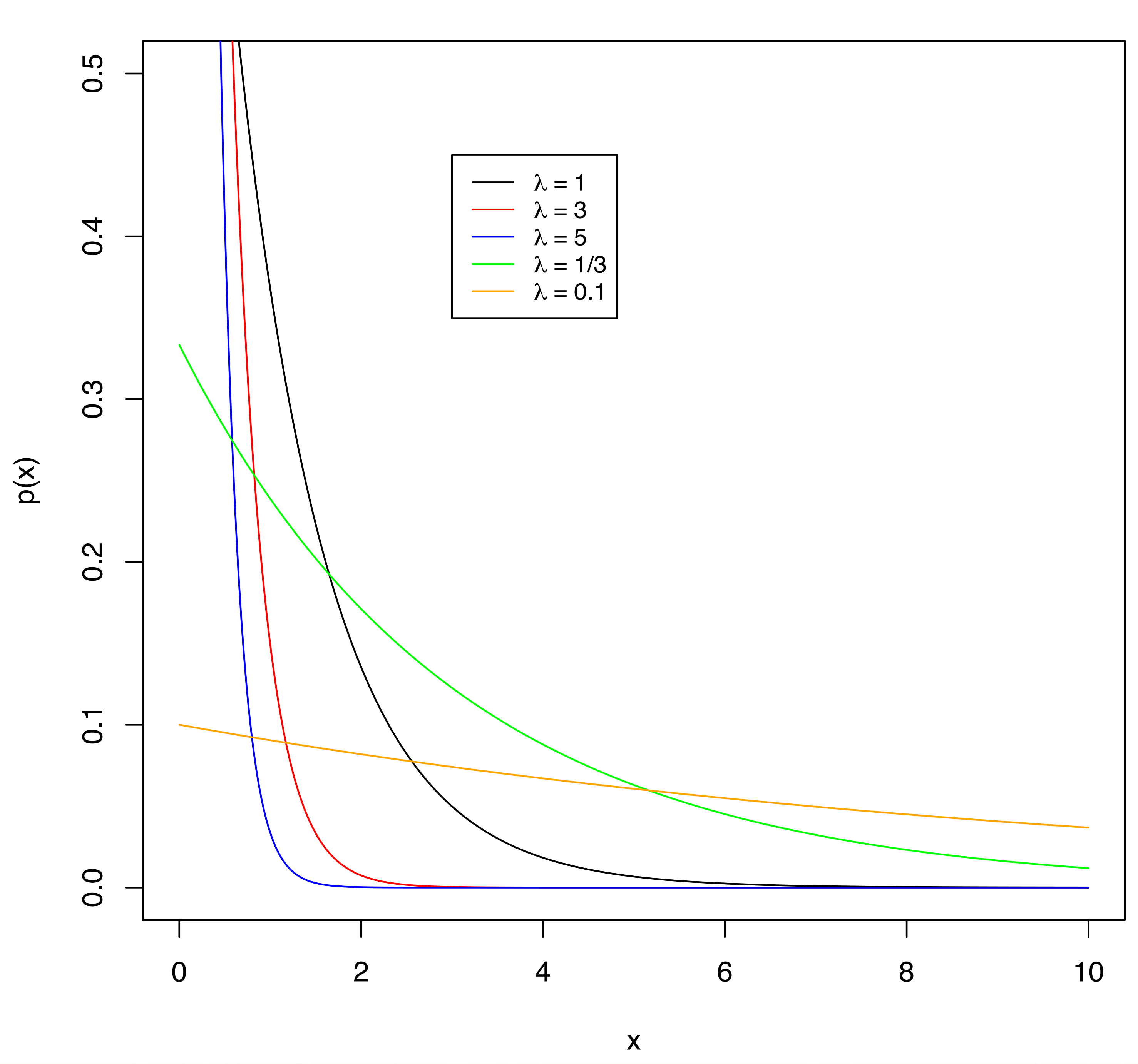


Continuous RVs: Exponential distribution

- The **Exponential distribution** depends on a *rate* parameter $\lambda > 0$

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- This arises as the distribution of the separation of events in a Poisson process.

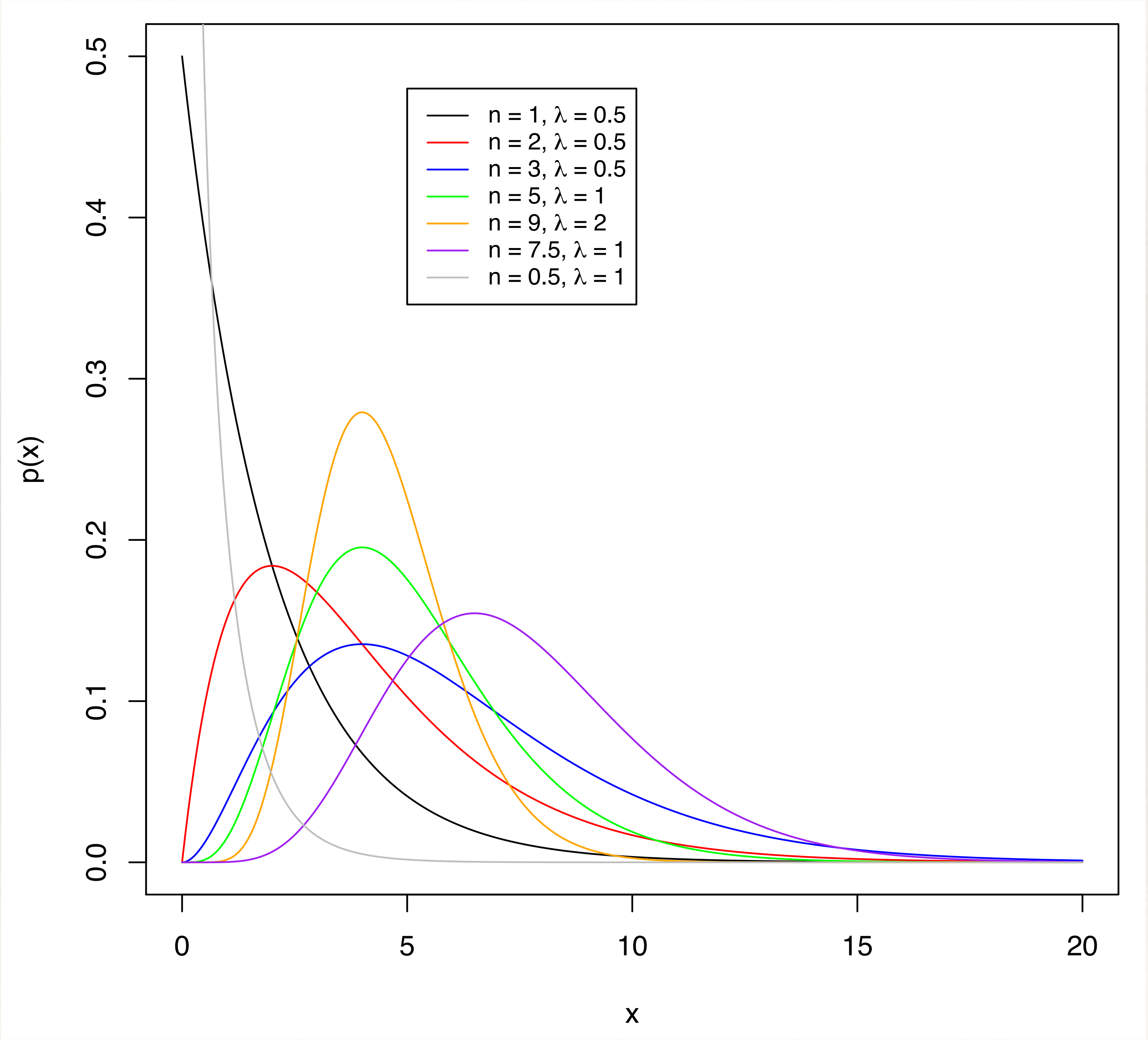


Continuous RVs: Gamma distribution

- The **Gamma distribution** depends on a *shape parameter* n > 0 and a *scale parameter* $\lambda > 0$

$$p(x) = \begin{cases} \frac{1}{\Gamma(n)} \lambda^n x^{n-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- The Gamma distribution is commonly used in Bayesian inference as a prior with support on the positive real line, and is conjugate to the Poisson distribution.



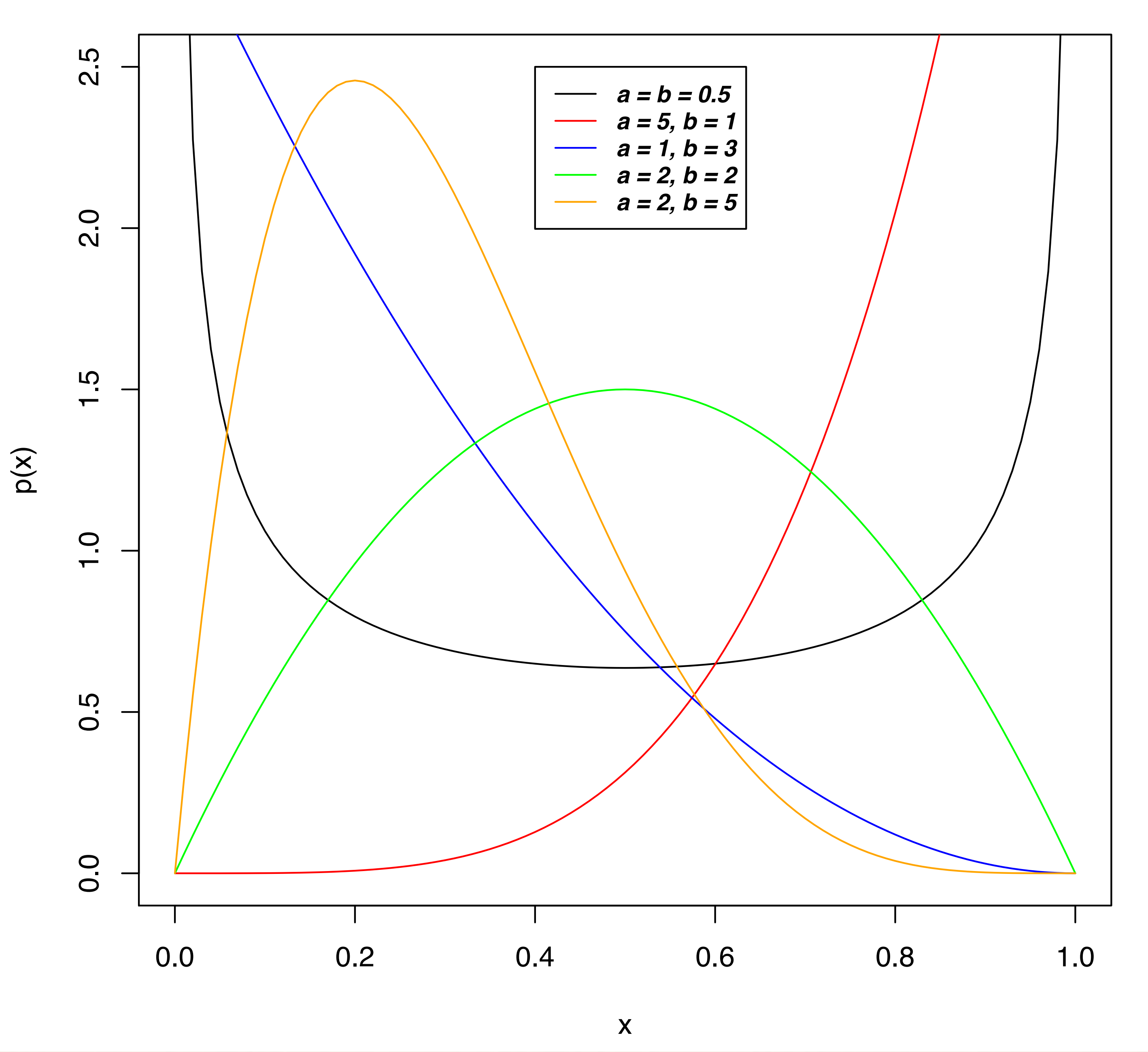
Continuous RVs: Beta distribution

- The **Beta distribution** depends on two *shape parameters* a, b > 0

$$p(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$$

- The Beta distribution is conjugate to the Binomial distribution and is used as a prior for parameters with support in [0,1].



Continuous RVs: Dirichlet distribution

- The **Dirichlet distribution** is a *multivariate distribution*, generating K samples $\{x_i\}$ constrained such that $0 < x_i < 1$ and

$$\sum_{i=1}^{K} x_i = 1$$

- The distribution depends on a vector of *concentration parameters*

$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_K)$$

- and has pdf

$$p(x) = \frac{1}{B(\vec{\alpha})} \prod_{i=1}^{K} x_i^{\alpha_i - 1}, \qquad \text{where } B(\vec{\alpha}) = \frac{\prod_{i=1}^{K} \Gamma(\alpha_i)}{\Gamma\left(\sum_{j=1}^{K} \alpha_j\right)}$$

- The **Dirichlet process** is used as a prior on probability distributions in Bayesian nonparametric inference.

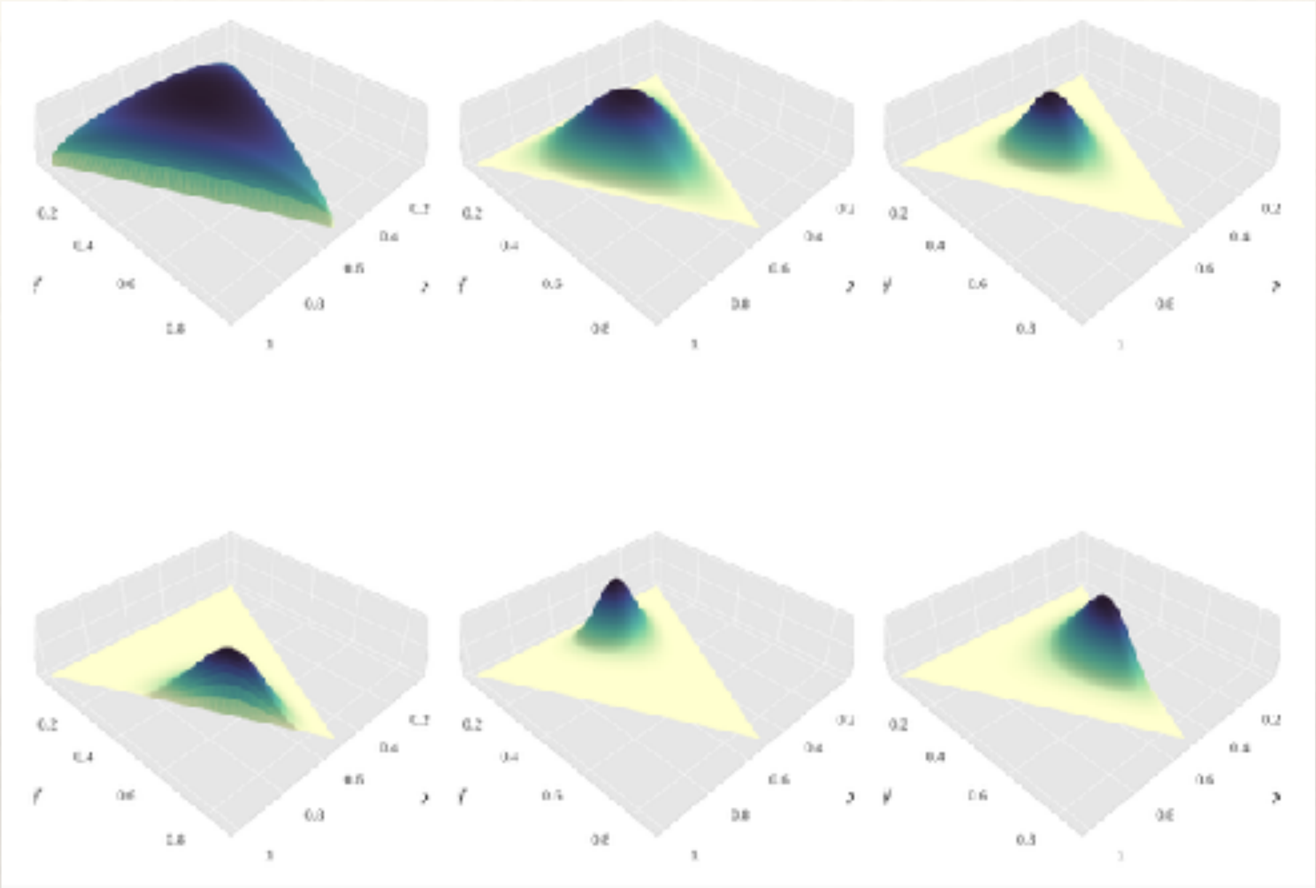


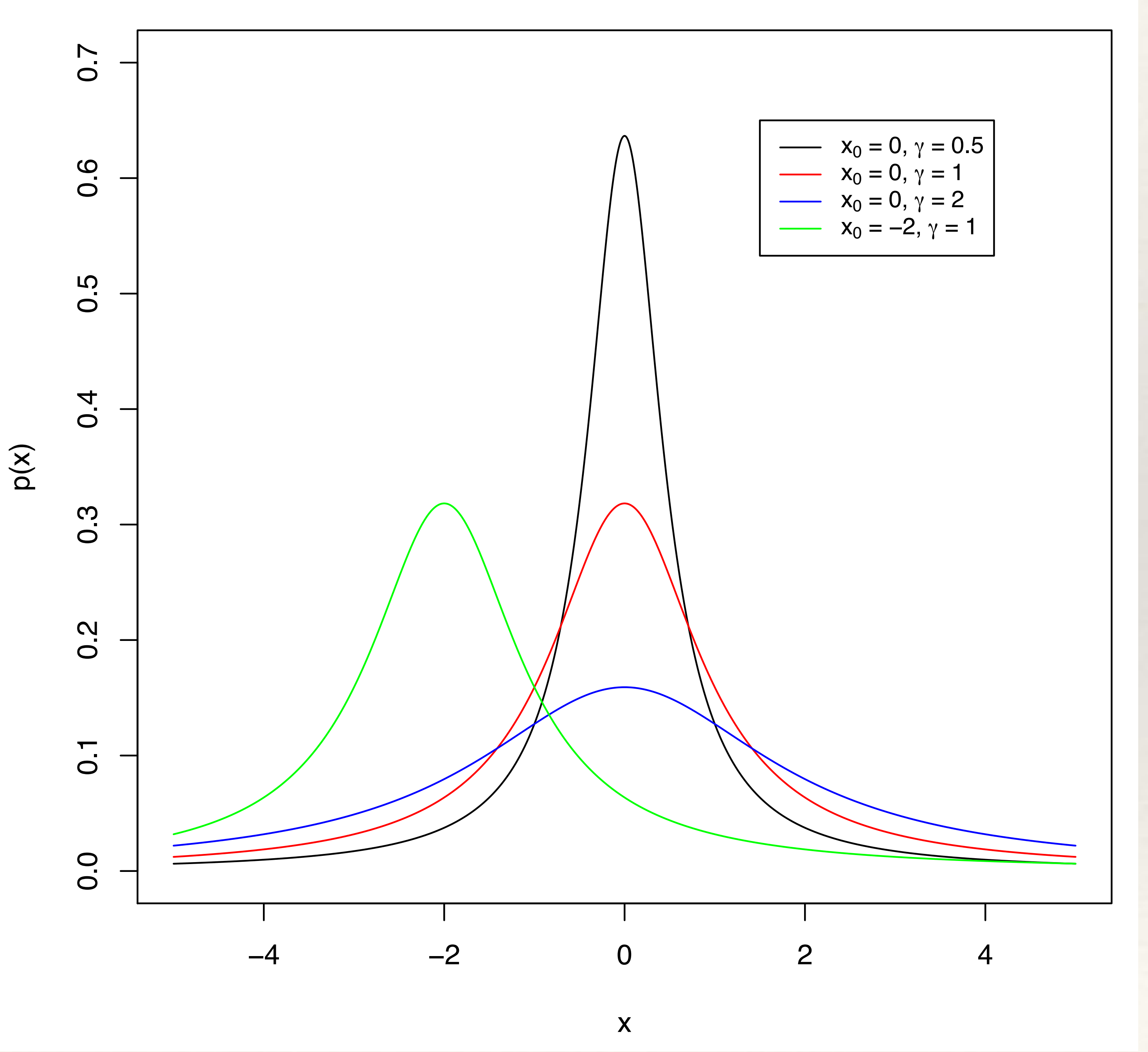
Figure from *Wikipedia*

Continuous RVs: Cauchy distribution

- The **Cauchy distribution** (or Lorentz distribution) depends on a *location parameter*, x_0, and a *scale parameter*, $\gamma > 0$

$$p(x) = \frac{1}{\pi\gamma\left[1 + \left(\frac{x - x_0}{\gamma}\right)^2\right]}$$

- This distribution arises in optics and is used to model distributions with sharp features, e.g., spectral lines in LIGO.



Summarising random variables: average

- The pdf (or pmf) completely characterises a probability distribution, but it is often more convenient to work with summary quantities.
- These are based on *expectation values*

$$\mathbb{E}(T(X)) = \int_{-\infty}^{\infty} p(x)t(x)\mathrm{d}x$$

- There are various quantities that summarise the *average* value of a random variable
  - **Mean**

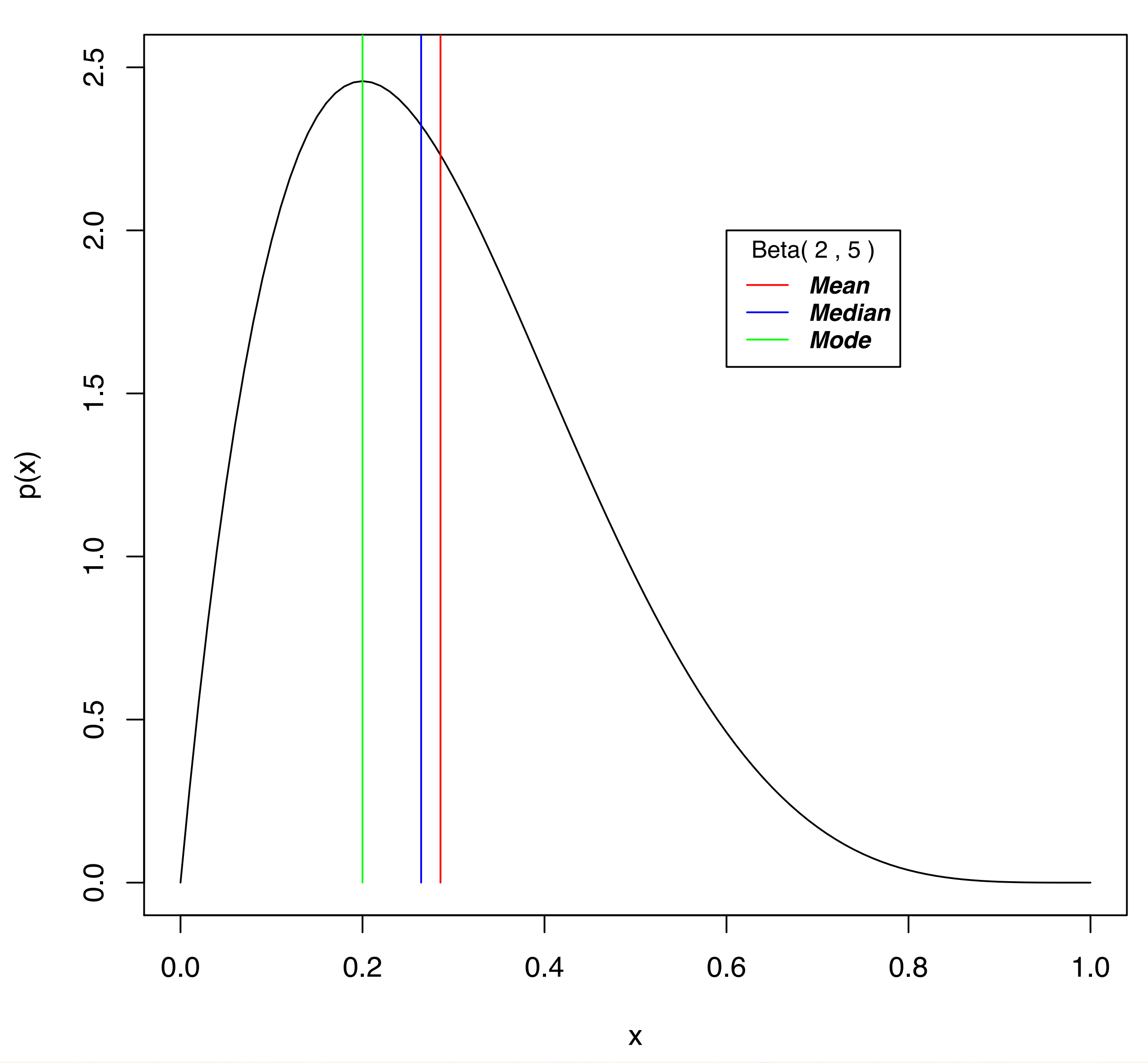
$$\mu = \mathbb{E}(X)$$

  - **Median** m satisfies

$$\int_{-\infty}^{m} p(x)\mathrm{d}x = \int_{m}^{\infty} p(x)\mathrm{d}x = \frac{1}{2}$$

  - **Mode**

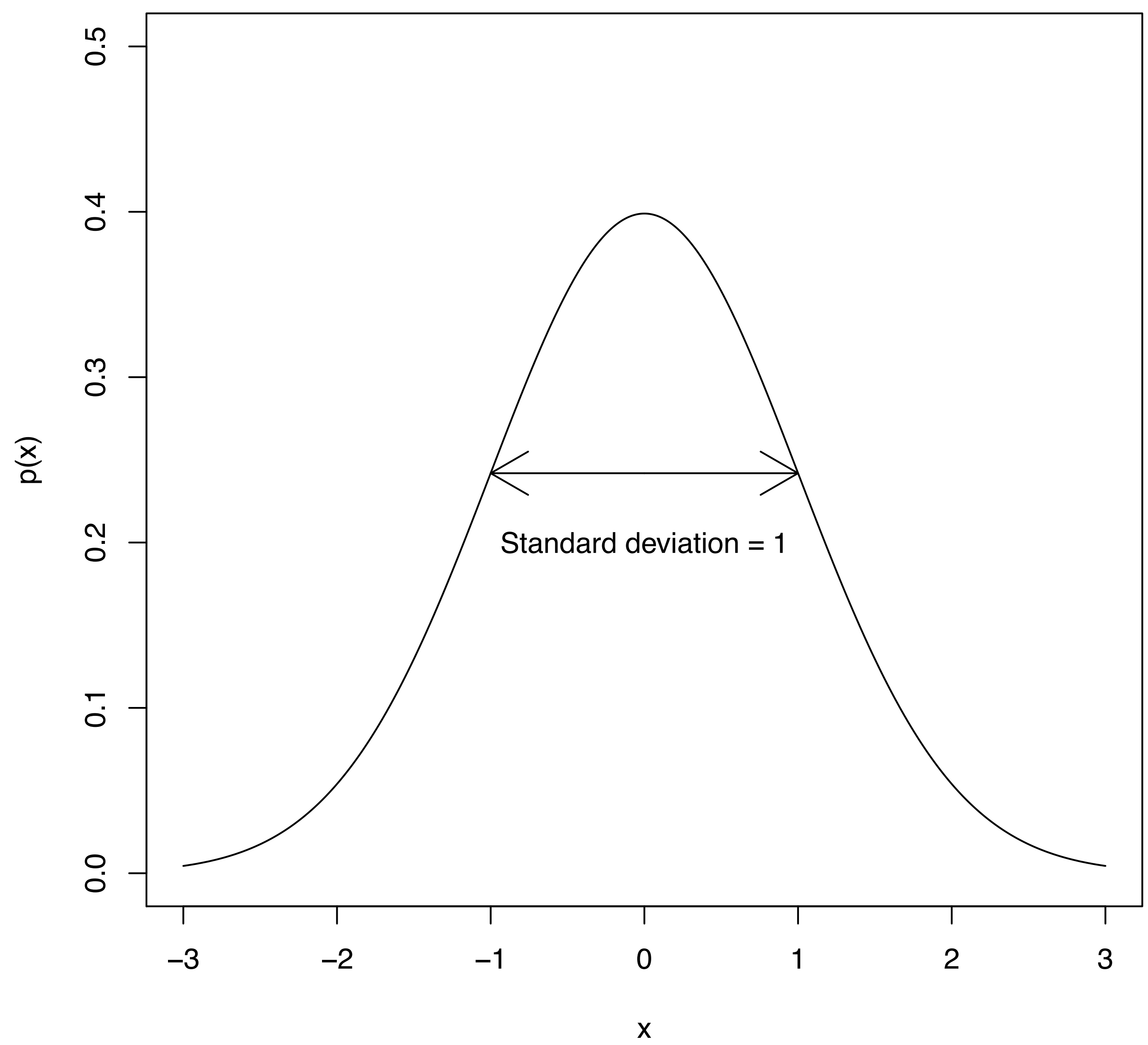
$$M = \mathrm{argmax}_{x\in\mathcal{X}} p(x)$$



Summarising random variables: spread

- Other quantities summarise the spread of a RV
  - **Variance/Standard deviation**

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right]$$



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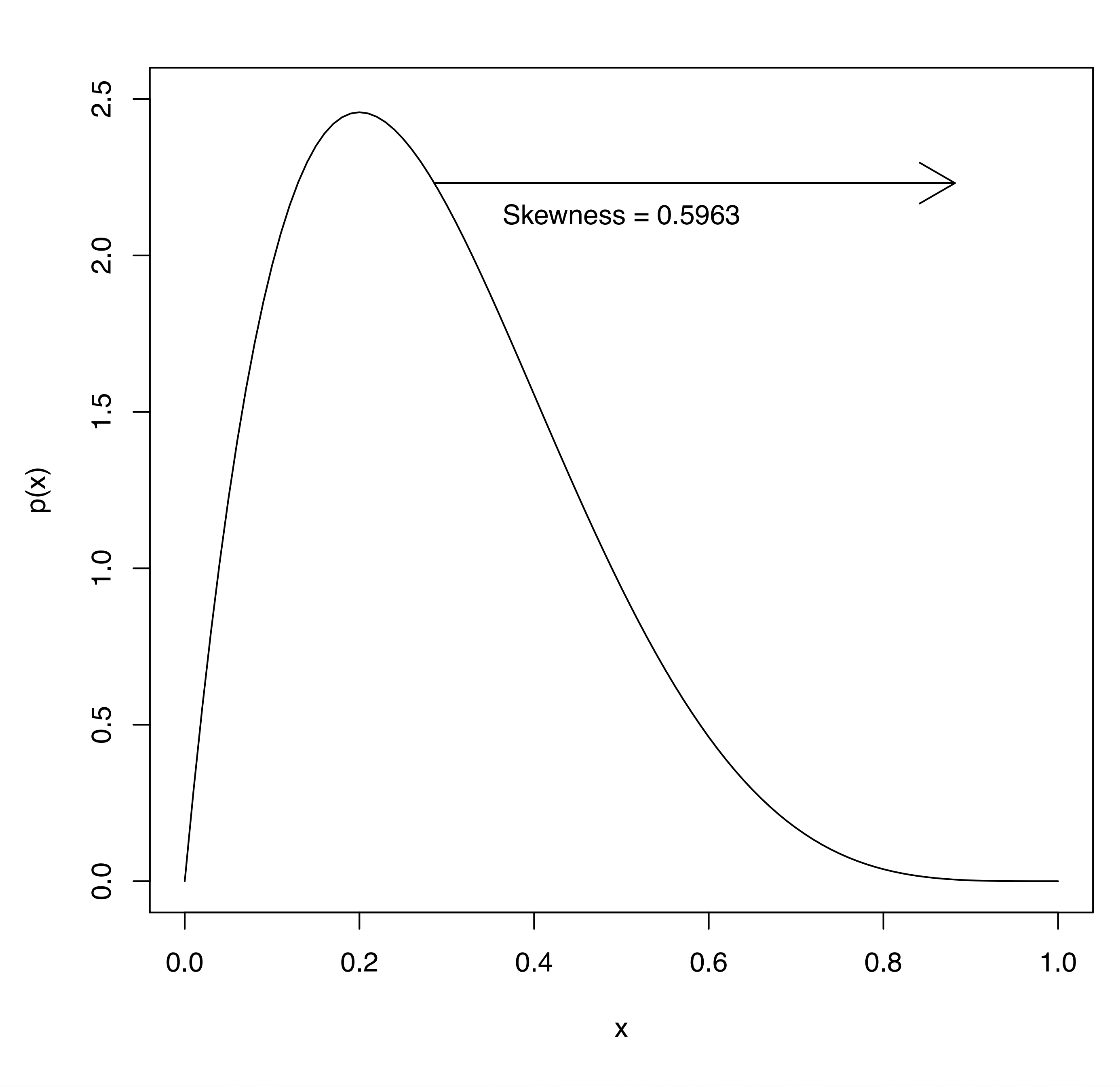
  - **Skewness**

$$\gamma_1 = \mathbb{E}\left[\left(\frac{x-\mu}{\sigma}\right)^3\right]$$



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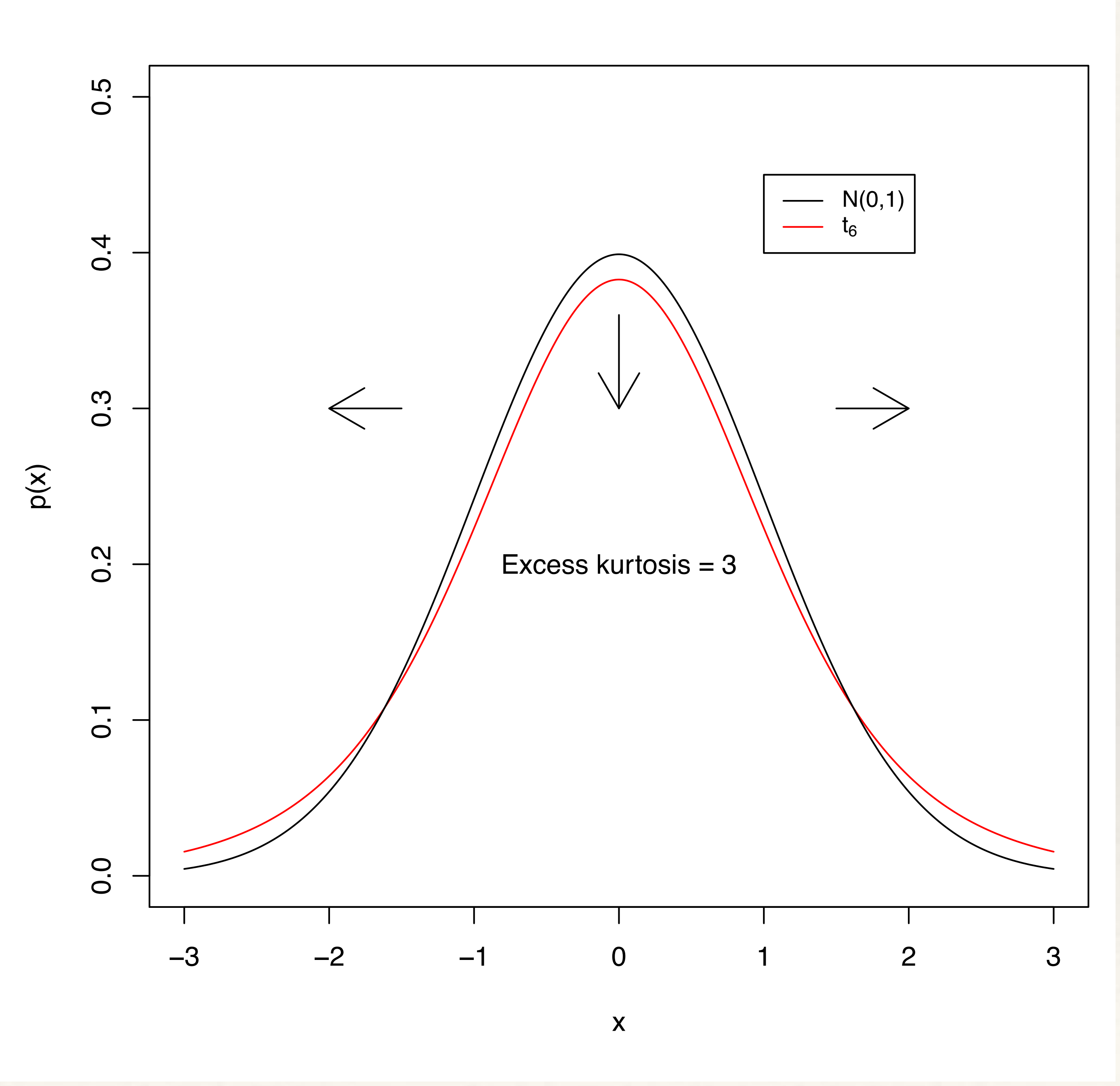
  - **Excess Kurtosis**

$$\mathrm{Kurt}(X) = \mathbb{E}\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] - 3$$



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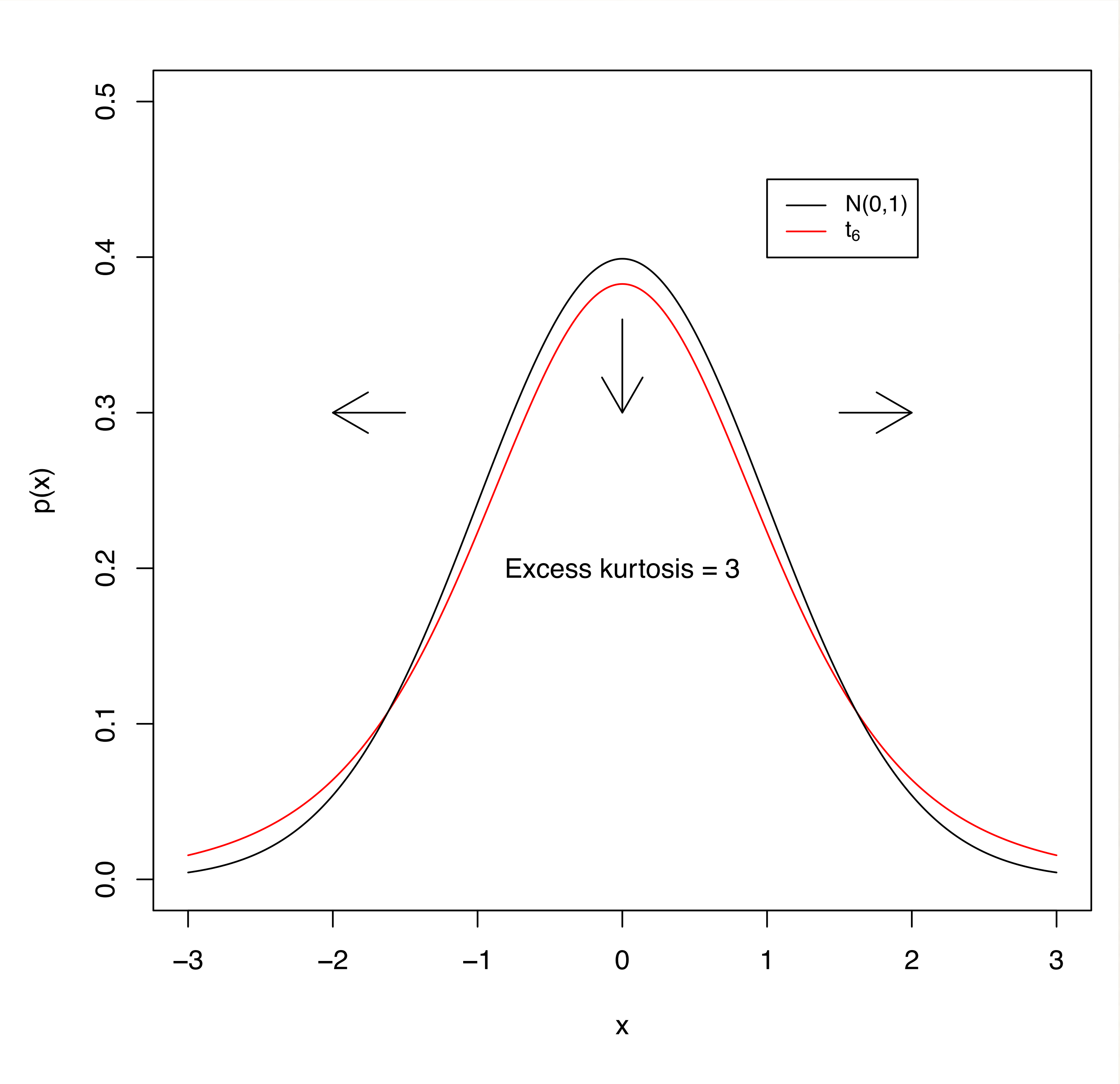
  - **Higher moments**

$$\mathbb{E}\left[(X-c)^n\right]$$

- Moments can be efficiently computed using the *moment generating function*

$$M_X(t) = \mathbb{E}\left[\mathrm{e}^{tX}\right] \quad t \in \mathbb{R}$$



Independence

- A set of random variables *{X_1, X_2, …, X_N}* is *independent* if, for all choices of

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_N \leq x_N) = P(X_1 \leq x_1)P(X_1 \leq x_1)\ldots P(X_1 \leq x_1)$$

- In terms of the density function this is equivalent to

$$p(x_1, \ldots, x_N) = p_{X_1}(x_1)p_{X_2}(x_2)\ldots p_{X_N}(x_N)$$

- Two independent random variables have zero covariance

$$\mathrm{cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}(X))\,(Y - \mathbb{E}(Y))\right] = 0$$

- but the converse is not necessarily true.
- Random variables are *independent identically distributed* (IID) if they are independent and are all drawn from the same probability distribution.

Linear combinations of RVs

- Suppose $X_1, \ldots, X_N$ are random variables and consider a new RV

$$Y = \sum_{i=1}^{N} a_i X_i$$

- Y has the properties

$$\mathbb{E}(Y) = \sum_{i=1}^{N} a_i \mathbb{E}(X_i), \qquad \mathrm{Var}(Y) = \sum_{i=1}^{N} a_i^2 \mathrm{Var}(X_i) + \sum_{i \neq j} a_i a_j \mathrm{cov}(X_i, X_j)$$

- The first equation holds for any random variables. If the RVs are *independent* then the relationships simplify

$$\mathrm{Var}(Y) = \sum_{i=1}^{N} a_i^2 \mathrm{Var}(X_i) \qquad M_Y(t) = \prod_{i=1}^{N} M_{X_i}(a_i t)$$

- If $\{X_i\}$ are IID then the *sample mean* defined by $a_i = 1/N$ for all i has the properties

$$\mathbb{E}(\hat{\mu}) = \mathbb{E}(X_1), \qquad \mathrm{Var}(\hat{\mu}) = \frac{1}{n}\mathrm{Var}(X_1), \qquad M_{\hat{\mu}}(t) = \left(M_{X_1}\left(\frac{t}{N}\right)\right)^N$$

Laws of large numbers

- Averages of random variables have various nice asymptotic properties

$$S_n = \sum_{i=1}^{n} X_i \qquad \mathbb{E}(X) = \mu \qquad \mathrm{Var}(X) = \sigma^2$$

- *Weak law of large numbers*: for $\epsilon > 0$

$$P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \to 0, \text{ as } n \to \infty$$

- *Strong law of large numbers*

$$P\left(\frac{S_n}{n} \to \mu\right) = 1$$

- *Central Limit Theorem*: for $S_n^* = \dfrac{S_n - n\mu}{\sigma\sqrt{n}}$

$$\lim_{n\to\infty} P(a \leq S_n^* \leq b) = \Phi(b) - \Phi(a)$$