Exercises

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Exercise 1.1

The mass function of a binary pulsar system is defined by:

$$f \equiv \frac{(M_2 \, \sin i)^3}{(M_{\rm psr} + M_2)^2} \tag{1.1}$$

where $M_{\rm psr}$ *is the pulsar mass,* M_2 *is the mass of its companion star, and* i *is the orbital inclination angle of the binary system (*$i = 90^\circ$ *for an edge-on geometry,* $i = 0^\circ$ *for a face-on geometry.*

a) *Show that the mass function can be written in terms of observables as:*

$$f = \frac{P_{\rm orb} \, v_{\rm psr}^3}{2\pi \, G} = \frac{4\pi^2}{G} \, \frac{(a_{\rm psr} \sin i)^3}{P_{\rm orb}^2} \tag{1.2}$$

where $v_{\rm psr} = 2\pi \, a_{\rm psr} \sin i / P_{\rm orb}$ *is the projected orbital velocity of the pulsar (parallel to the line of sight),* $a_{\rm psr} \sin i$ *is the projected semi-major axis of the pulsar orbit around the common centre of mass, and* $P_{\rm orb}$ *is the orbital period of the binary given by Kepler's 3rd law.*

b) *Consider a binary MSP with an orbital period of* $P_{\rm orb} = 2.00 \ {\rm d}$ *and a projected semi-major axis of* $a_{\rm psr} \sin i = 1.90 \ {\rm lt \cdot s}$*. Determine its mass function.*

c) *Assume a NS mass of* $1.30 \ M_\odot$ *or* $1.80 \ M_\odot$ *and plot the mass of its companion star as a function of the unknown orbital inclination angle,* i*. What is the expected nature of the companion star?*

Exercise 1.2

The orbital angular momentum of a binary is given by:

$$J_{\rm orb} = \frac{M_1 M_2}{M_1 + M_2} \, \Omega \, a^2 \, \sqrt{1 - e^2} \tag{1.3}$$

Combining the above expression with Kepler's third law yields the following expression for the orbital angular momentum:

$$J_{\rm orb} = \sqrt{\frac{G \, M_1^2 M_2^2 \, a(1 - e^2)}{M_1 + M_2}} \tag{1.4}$$

The relative rate of change of the orbital angular momentum in a circular binary ($e = 0$) due to emission of GWs is given by:

$$\frac{\dot{J}_{\rm GW}}{J_{\rm orb}} = -\frac{32G^3}{5c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^4} \tag{1.5}$$

Show that this equation can be combined with the above expressions and integrated to reveal a merger timescale:

$$\tau_{\rm GW} = \frac{1}{8}\left|\frac{J}{\dot{J}}\right| \tag{1.6}$$

Exercise 1.3

Consult the literature for calculating the GW merger time of a given binary system. Notice that a numerical integration is needed for an eccentric binary (see e.g. Section 7 in the paper cited below in Exercise 1.4).

Calculate the GW merger time of the Hulse-Taylor pulsar (PSR B1913+16 with $P_{\rm orb} = 7.75$ hr, $e = 0.617$ and $M_{\rm NS} = 1.441$ and $1.387\,M_\odot$) and compare to a system, with similar NS masses and $P_{\rm orb}$, in a circular orbit ($e = 0$) and a system with $e = 0.90$.

Exercise 1.4

Kinematic impact of the 2nd SN in producing a BH-BH system, see e.g. Sections 6.1–6.3 in Tauris et al. (2017), ApJ 846, 170:
https://ui.adsabs.harvard.edu/abs/2017ApJ...846..170T

Consider a pre-SN system composed of an exploding WR star with a mass of $M_{\rm He} = 10.0\,M_\odot$ in orbit with the first-formed BH with a mass of $M_2 = 12.0\,M_\odot$ and an orbital period of $P_{\rm orb} = 3.00$ d. Assume that the asymmetric SN leaves behind a second BH with a mass of $M_{\rm BH} = 7.0\,M_\odot$ and also imparts a kick onto it with a magnitude of $w = 200\,{\rm km\,s^{-1}}$ in the direction: $\theta = 114^\circ$ and $\phi = 83^\circ$.

Derive for the post-SN binary:

- *a) the critical angle, $\theta_{\rm crit}$ for which $\theta < \theta_{\rm crit}$ would have disrupted the binary, and verify thereby that this system will remain bound.*
- *b) the ratio $(a_{\rm f}/a_{\rm i})$ of the post- to pre-SN orbital semi-major axis.*
- *c) the orbital period, $P_{\rm orb}$.*
- *d) the eccentricity, e.*

e) the 3D systemic recoil velocity of the system, $v_{\rm sys}$*.*

f) the misalignment angle, δ*.*

g) keeping the kick direction the same, what is the maximum kick velocity imparted onto the BH such that the post-SN system will remain bound.

Exercise 1.5

The dimensionless GW amplitude generated by a binary at luminosity distance, d_L *for an average orbital orientation and polarization is given by:*

$$h = \left(\frac{32}{5}\right)^{1/2} \frac{\pi^{2/3} G^{5/3} f_{\rm GW}^{2/3} \mathcal{M}^{5/3}}{c^4 \, d_L} \tag{1.7}$$

Show that the GW amplitude, h is dimensionless.
(Can you derive this equation?)

Exercise 1.6

Show that for a symmetric SN, the changes in semi-major axis and eccentricity are given by:

$$\frac{a}{a_0} = \frac{M}{M - \Delta M} \qquad\qquad e = \frac{\Delta M}{M} \tag{1.8}$$

where a *and* a_0 *denote the post- and pre-SN semi-major axis, respectively,* M *is the post-SN total mass of the binary and* ΔM *is the amount of mass lost in the SN. Assume the explosion to be instantaneous and that the pre-SN orbit is circular.*

(Hint: consider the ratio of the orbital (total) energy before and after the SN, where $E_{\rm orb} = E_{\rm pot} + E_{\rm kin}$*, and where* $E_{\rm orb} = -GM_1M_2/2a$*,* $E_{\rm pot} = -GM_1M_2/r$ *and* $E_{\rm kin} = 1/2\,\mu\, v_{\rm rel}^2$*. A circular pre-SN orbit means that* $a_0 = r$*. Notice, after the SN, the periastron separation of the new orbit,* $q = a(1-e)$ *is equal to the orbital separation between the two stars at the moment of the explosion (*$r = a_0$*).)*

Answers to Exercises

A few numerical answers to selected exercises. For solutions to derivations of equations, general discussions etc., please consult your course instructor.

Exercise 1.1: b) $f = 1.85 \times 10^{-3}\ M_{\odot}$.

Exercise 1.3: $\tau_{\rm GW} = 301$ Myr (1.65 Gyr for $e = 0$. 5.57 Myr for $e = 0.90$).

Exercise 1.4: a) $\theta_{\rm crit} = 59.31°$. b) $(a_{\rm f}/a_{\rm i}) = 0.9739$. c) $P_{\rm orb} = 3.102$ d. d) $e = 0.06454$. e) $v_{\rm sys} = 93.99\ {\rm km\,s^{-1}}$. f) $\delta = 28.57°$. g) $w_{\rm max} = 559.1\ {\rm km\,s^{-1}}$.