Computational physics: Tutorial 3

Problems with a $*$ are hard and optional.

Problem 1 *Harmonic coordinates*

1. Consider a FRW solution:

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right) \ . \tag{1}$$

Compute $\Box x^a$ for this metric. Transform to a new time coordinate $\tau(t)$ such that the coordinates are harmonic, i.e. $\Box x^a = 0$. For a de Sitter solution with

$$\left(\frac{da/dt}{a}\right)^2 = \frac{\Lambda}{3} \tag{2}$$

what is the explicit relation between τ and t?

2. Start with the metric of a Schwarzschild black hole of mass M and radius r:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_{\mathrm{SCHW}}^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3}$$

. Consider a coordinate transformation

$$\begin{aligned} t &= t_{\mathrm{SCHW}} + 2M \log\left|\frac{r - 2M}{r}\right| , \\ x + iy &= (r - M) e^{i\phi} \sin\theta \ , \\ z &= (r - M) \cos\theta \ . \end{aligned} \tag{4}$$

Show that these new coordinates are harmonic. Where is the black hole horizon in these new coordinates? Are these coordinates horizon penetrating? Explain. Hint: Examine the behavior of the lapse on the horizon.

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Problem 2 *3+1 Decomposition of Maxwell Equations*

In this question we derive the 3+1 decomposition of the Maxwell equations. As in class, we assume the spacetime is foliated with spacelike hypersurfaces with unit normal

$$n^a = \alpha^{-1}(\partial_t^a - \beta^a).$$

- First express the Lie derivative along the evolution vector αn^a of a vector field E^a in terms of (covariant) derivatives of E^a, n^a, and α.

  In Vacuum, the Maxwell equations are given by

  $$\nabla_a F^{ab} = 0, \text{ and, } \nabla_b \star F^{bd} = 0 \tag{5}$$

  where $\star F^{ab} = \frac{1}{2}\epsilon^{abcd}F_{cd}$ is dual tensor.

- Using $E^a = F^{ab}n_b$, $B^a = \star F^{ab}n_b$, express in 3+1 form the evolution equation for E^a and its associated constraint.
  Hint: start by decomposing the Faraday tensor F^{ab} into electric and magnetic parts as seen by Eulerian observer n^a using the definitions given above. Also remember that E^a and B^a are purely spatial vectors i.e. $E^a n_a = B^a n_a = 0$.

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Problem 3 *BSSN equations*

In this question, we will derive a commonly used formulation of the Einstein equations in numerical relativity known as the BSSN (Baumgarte-Shapiro-Shibata-Nakamura) formulation. This involves a conformal decomposition of the three metric γ_{ij} and extrinsic curvature K_{ij}. In particular, we decompose γ_{ij} into a conformal factor and conformal metric:

$$\gamma_{ij} = \tilde{\gamma}_{ij} e^{4\phi} \ , \tag{6}$$

where $\det \tilde{\gamma}_{ij} = 1$. We also decompose K_{ij} into its trace $K = \gamma_{ij}K^{ij}$, and a conformal trace-free part

$$\tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) = e^{-4\phi}K_{ij}^{TF} \tag{7}$$

(we denote the trace-free part of a tensor by TF). Note that we will take the convention that the indices of $\tilde{A}_{ij}$ are raised by $\tilde{\gamma}_{ij}$.

Beginning with the 3+1 decomposition of the Einstein equations for γ_{ij} and K_{ij} for a vacuum spacetime as presented in class, we will now derive the BSSN evolution equations.

1. By decomposing the 3+1 evolution equations for the metric into trace and trace-free parts, show that three-metric components satisfy the following equations

$$(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} \tag{8}$$

and

$$(\partial_t - \mathcal{L}_\beta)\phi = -\frac{1}{6}\alpha K \tag{9}$$

where $\mathcal{L}_\beta$ denotes the Lie derivative with respect to the shift vector.

2. Similarly for the extrinsic curvature decomposition, show that

$$(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} = e^{-4\phi}\left(\alpha R_{ij}^{TF} - (D_i D_j \alpha)^{TF}\right) + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) \tag{10}$$

and

$$(\partial_t - \mathcal{L}_\beta)K = -D^i D_i \alpha + \alpha \tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}\alpha K^2 \ . \tag{11}$$

The Ricci tensor can also decomposed into contributions from the conformal factor and metric:

$$\begin{aligned} R_{ij} = & -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{\gamma}_{ij}(\tilde{D}^k\phi)(\tilde{D}_k\phi) \\ & - \frac{1}{2}\tilde{\gamma}^{kl}\partial_k\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\left(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}\right) \ . \end{aligned} \tag{12}$$

(You don't have to derive this.) Here $\tilde{\Gamma}^i = \tilde{\gamma}_{ij}\tilde{\Gamma}^i_{jk}$ is the conformal connection function constructed from the conformal connection coefficient $\tilde{\Gamma}^i_{jk}$ associated with $\tilde{\gamma}_{ij}$, and $\tilde{D}_i$ is the associated derivative operator.

Explain how in this form, the contribution from conformal metric (second line) has a principal part (first term) that is manifestly elliptic, since it is just a coordinate derivative Laplace operator acting on the components of the conformal metric.

3. Show that the evolution equation of $\tilde{\Gamma}^i$ is given by

$$\partial_t\tilde{\Gamma}^i = -\partial_j\left(2\alpha\tilde{A}^{ij} - 2\tilde{\gamma}^{k(j}\partial_k\beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_k\beta^k + \beta^k\partial_k\tilde{\gamma}^{ij}\right) \ . \tag{13}$$

In the BSSN formulation, the evolution variables are ϕ, $\tilde{\gamma}$, K, $\tilde{A}_{ij}$, and $\tilde{\Gamma}^i$, along with appropriate gauge conditions for the lapse and shift. Check that this is the same number of degrees of freedom as the ADM or harmonic evolution equations. The BSSN equations have improved hyperbolicity properties compared to the ADM equations, see e.g. Sarabach et al. 2002.

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Problem 4 ⋆ *Hamiltonian Constraint* ⋆

Let's consider an initial timeslice with energy density ρ and momentum density j^i, where we require the trace of the extrinsic curvature to be a constant K, and the trace-free part of the extrinsic curvature to be zero.

1. Write down the simplified Hamiltonian and momentum constraints under these assumptions.

2. Comment on the number of remaining metric degrees of freedom for data on the timeslice, and suggest in words one strategy for specifying the remaining free data while finding a solution to the constraints.

3. Restrict to the case where the timeslice is given by a cube with periodic boundary conditions, or equivalently, a three-torus. Assume that on the timeslice, the volume integral of the spatial curvature R satisfies

$$\int R\sqrt{\gamma}\, dV \leq 0$$

(here $\gamma = \det \gamma_{ij}$ just gives the proper spatial volume element). Derive a lower bound on $|K|$, and comment what this means for the possibility of such spacetimes being stationary.

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Problem 5 ⋆ *Harmonic Formulation of GR* ⋆

The harmonic equations of motion (in vacuum) are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} - \frac{1}{2}\left(\nabla_a\Gamma_b + \nabla_b\Gamma_a\right) + \frac{1}{2}g_{ab}g^{cd}\nabla_c\Gamma_d = 0$$

with the constraint that the harmonic condition

$$\Gamma_a := g_{ab}g^{cd}\Gamma^b_{cd} = 0$$

is satisfied.

1. Given data $g_{ab}(t = 0)$ and $\partial_t g_{ij}(t = 0)$ on some initial timeslice, where the indices (i, j) only run over the spatial components, show that the four components $\partial_t g_{ta}$ can always be chosen such that the harmonic condition is satisfied.

2. Does the choice of the components $\partial_t g_{ta}$ affect whether the Hamiltonian and momentum constraints are satisfied by the initial data? Explain.

3. Derive an evolution equation for Γ_a. Hence, show that if the harmonic condition is satisfied at $t = 0$, it will continue to be satisfied under evolution with the harmonic equations of motion, assuming the Hamiltonian and momentum constraints are also satisfied at $t = 0$.

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