Generalized harmonic formulation

Recap

* 3+1 decomposition of EEs

$\rightarrow (\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$

$\rightarrow (\partial_t - \mathcal{L}_\beta)K_{ij} = -D_i D_j \alpha + \alpha\{ {}^{(3)}K_{ij} + KK_{ij} - 2K_{ik}K^k{}_j + 4\pi[(S-E)\gamma_{ij} - 2S_{ij}]\}$

evolution, hyperbolic

$\rightarrow {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$

$\rightarrow D_j K^j{}_i - D_i K = 8\pi p_i$

constraint elliptic

* Well-posedness

$$\|\vec{u}(t)\| \leq K e^{\alpha t}\|\vec{u}(t=0)\|$$

K, α independent of $\vec{u}(t=0)$

$\Leftrightarrow$ strongly hyperbolic : principal part has real eigenvalues + complete set of eigenvectors

Coordinate / gauge freedom

Bad time slicings, an example (see Carroll, App. F)

e.g. Gaussian normal coordinates

$\alpha = 1, \ \beta^i = 0 \quad \Leftrightarrow \quad g_{tt} = -1, \ g_{ti} = 0$

$\Rightarrow n^a = (1, \vec{0})$ geodesic

$n^a \nabla_a n^b = -\Gamma^b{}_{tt} = 0$

$$K = g^{ab}K_{ab} = -g^{ab}(g^c{}_a + n_a n^c)\nabla_c n_b$$

$$= -\nabla_a n^a \equiv -\theta$$ expansion of congruence of geodesics

$\rightarrow$ Raychaudhuri eqn

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 + \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} = -R_{ab}u^a u^b$$

σ_{ab} is shear

$\omega_{ab} = \frac{1}{2}(\nabla_a u_b - \nabla_b u_a)$ rotation tensor

↳ $\omega_{ab}\omega^{ab} \geq 0$ (spatial tensor)

$R_{ab}u^a u^b = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)u^a u^b \geq 0$ strong energy condition

$T_{ab}t^a t^b \geq \frac{1}{2} T t^a t_a$

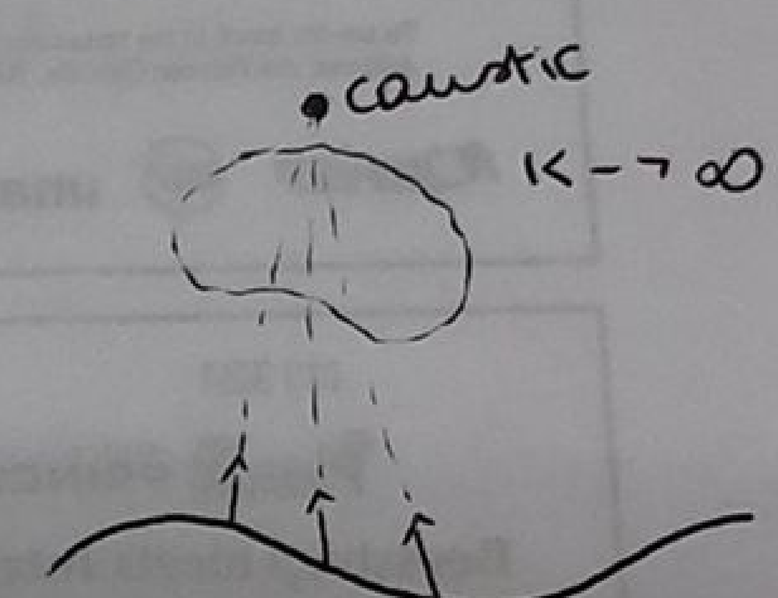
$$\Rightarrow \frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 - \omega_{ab}\omega^{ab} \leq 0$$

For $u^a = n^a$: $\theta = -K$, $\omega_{ab} = \frac{1}{2}(\partial_a n_b - \partial_b n_a) = 0$

$$-\frac{dK}{d\tau} + \frac{1}{3}K^2 \leq 0$$

if $K(\tau = 0) = K_0 > 0$: $\int_{K_0}^{K} \left(-\frac{1}{K^2}\right) dK \leq \int_0^{\tau} -\frac{1}{3}\, d\tau'$

$$K^{-1} \leq K_0^{-1} - \frac{\tau}{3}$$



$\rightarrow K^{-1} > 0$ at $\tau = 0$

$K^{-1} < 0$ at $\tau > 3K_0^{-1}$

GH formulation of the EEs

Fix gauge DOF : $\underset{\nabla_a\nabla^a}{\square} x^a = H^a$

H^a source fction (not vector)

trace-reversed EE

$$R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)$$

$$= \overbrace{-\frac{1}{2} g^{cd}\partial_c\partial_d g_{ab}}^{\text{wave eqn}} - \underbrace{\partial_c g_{d(a}\partial_{b)} g^{cd}}_{} + \nabla_{(a}\Gamma_{b)}$$

"non-hyperbolic" 2nd order terms

$$- \underbrace{\Gamma_c \Gamma^c{}_{ab} - \Gamma^c{}_{da}\Gamma^d{}_{cb}}_{}$$

lower order terms (1st der)

where $\Gamma_a = g^{bc}\Gamma_{abc} = -g_{ab}\nabla_c\nabla^c x^b = -H_a$

$$\nabla_c\nabla^c x^b = \frac{1}{\sqrt{-g}}\partial_c\left(\sqrt{-g}\, g^{cb}\right) = -\Gamma^b$$

Rewrite EEs using $H_a = -\Gamma_a$

$$-8\pi(2T_{ab} - g_{ab}T) = \underbrace{g^{cd}\partial_c\partial_d g_{ab}}_{\text{wave operator}} + F(g_{ab}, \partial_c g_{ab}, H_a, \partial_a H_b)$$ well-posed!

and

$$\underbrace{L(H^a)}_{\text{some evolution operator}} = 0 \quad \Rightarrow \text{Evolve } \{g_{ab}, \partial_t g_{ab}, H_a, \partial_t H_a\}$$

→ Simplest choice: $H^a = 0$ harmonic gauge

→ Simple choice: $H^a = F(g_{ab})$

$$\Leftrightarrow (\partial_t - \mathcal{L}_\beta)\alpha = -\alpha(H_t - \beta^i H_i + \alpha K)$$

$$\partial_t \beta^i = \beta^j \partial_j \beta^i + \alpha^2 \gamma^{ij} H_j + \alpha^2\, {}^{(3)}\Gamma^i_{jk}\gamma^{jk} - \alpha D^i \alpha$$

$H_{t,i} \approx$ time derivatives of lapse + shift

GH constraint evolution

$$C^a \equiv H^a - \Box x^a$$

Is $C^a = 0$ for all time?

$$R_{ab} - 4\pi(2T_{ab} - g_{ab}T) = \nabla_{(a}C_{b)}$$

$\hookrightarrow n^a n^b (\nabla_{(a}C_{b)}) =$ Hamiltonian constraint

$\hookrightarrow n^a \gamma^b{}_i (\nabla_{(a}C_{b)}) =$ Momentum constraint

$\Rightarrow \partial_t C_a = 0$ at $t=0$

↓ undo trace reversal

$$(R_{ab} - \tfrac{1}{2}R g_{ab}) - 8\pi T_{ab} = \nabla_{(a}C_{b)} - \tfrac{1}{2} g_{ab} g^{cd}\nabla_{(c}C_{d)}$$

↓ divergence

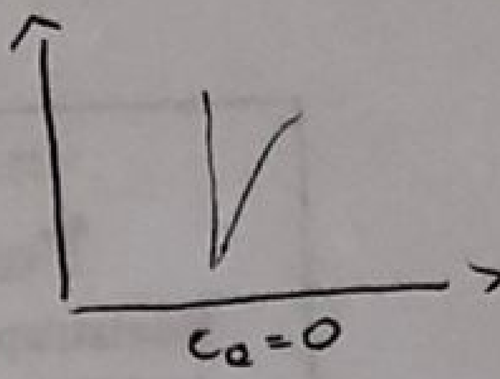
$$\underbrace{\nabla^a(R_{ab} - \tfrac{1}{2}R g_{ab})}_{0 \text{ Bianchi}} - \underbrace{(\nabla^a T_{ab})}_{0 \text{ SE cons.}} 8\pi = 0 = \nabla^a\nabla_a C_b + \nabla^a\nabla_b C_a - \nabla_b\nabla^a C_a$$

$$\Rightarrow \nabla_a\nabla^a C_b = -R^a{}_b C_a$$

$\Rightarrow$ if $C_a = \partial_t C_a = 0$ at $t=0$ $\Rightarrow C_a = 0$ for all time :)

Constraint damping

GH so far leads to fast growth of (initially) small C_a (expon)



Fix

→ Add $\kappa\,[\, n_{(a} C_{b)} - \frac{1}{2} g_{ab}\, n^c C_d \,]$ to EEs.

(no effect if $C_a = 0$)

⇒ same calc as above

$$\Box C^a = -R^a{}_b C^b + \kappa \nabla_b [\, n^{(b} C^{a)} \,]$$

$$\downarrow$$

$\kappa \sim \frac{1}{[L]}$ damps constraint violation on κ^{-1} length/timescale

Analogy with Maxwell eqns

GH form: $R_{ab} = 0 \rightarrow R_{ab} - \nabla_{(a} C_{b)} = 0$, $C^a = H^a - \Box x^a$

Maxwell: $\partial_a F^{ab} = \partial^a \partial_a A^b - \partial_a \partial^b A^a = 0$

$\partial^a F_{ab} = 0$

$F_{ab} = \partial_a A_b - \partial_b A_a$

time compon $b=0$

$$-\partial_t^2 A_t + \partial_i \partial^i A_t + \partial_t^2 A_t - \partial^i \partial_t A_i = 0$$

$$\partial^i (\partial_i A_t - \partial_t A_i) = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

But if choose Lorentz gauge

$$\partial_a A^a = 0$$

Maxwell eqns become ⇒ $\partial_a \partial^a A_b = 0$

$$\partial^b (\partial^a \partial_a A_b) = \partial_a \partial^a (\partial^b A_b) = 0$$