3+1 Decomposition of spacetime

References

- GR, Wald, Ch 12. Section 2 (broad picture)
- 3+1 formalism, Eric Gourgoulhon, 0703035 (very detailed)
- Intro to 3+1 NR, Alcubierre

Hypersurfaces of spacetime

4-dim. spacetime manifold $\mathcal{M}$

3-dim ~~spacetime~~ manifold $\hat{\Sigma}$

Embedding of $\hat{\Sigma}$ in $\mathcal{M}$

$\Phi : \hat{\Sigma} \to \mathcal{M}$, 1·1, Φ, Φ^{-1}

$\Sigma = \Phi(\hat{\Sigma})$ is called hypersurface

Level set $\quad \Sigma = \{ x^a \in \mathcal{M} \mid F(x^a) = 0 \}$

$\to \partial_a F$ is normal to Σ $\iff (\partial_a F) v^a = 0$ for v^a tangent to Σ

- $\partial_a F$ timelike, call Σ spacelike
- $\partial_a F$ spacelike, " " timelike
- $\partial_a F$ null, " " null

$\to$ restrict to Σ spacelike

Choose coordinates: $\quad x^a = (t, x^i)$ with x^i coordinates $\in \hat{\Sigma}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\llcorner$ 3-index

$$\to \Sigma = \{ x^a \in \mathcal{M} \mid t = 0 \}$$

$$\Phi : \hat{\Sigma} \to \mathcal{M}$$
$$x^i \to (0, x^i)$$

$\to$ unit normal to Σ: $\quad n_a = \dfrac{- \nabla_a t}{\sqrt{-\nabla_b t \nabla^b t}}$, $n_a n^a = -1$

$n^a = (1, \vec{0})$ in flat space (future pointing)

$\to \Phi : \hat{\Sigma} \to \mathcal{M}$

$v^i \to (0, v^i)$ $\quad$ push-forward mapping

$\Phi^* : \mathcal{M} \to \hat{\Sigma}$

$(u_t, u_i) \to u_i$ $\quad$ pull-back

inner product preserved by mapping

$$M \longrightarrow \hat{\Sigma}$$

$$(0, v^i)(u_t, u_i) = v^i u_i$$

For g_{ab} metric in M, induced metric γ_{ij} on Σ

$\gamma_{ij} = g_{ij}$ in adapted coord.

From now on $\Sigma \cong \hat{\Sigma}$

$$\rightarrow v^a = \underbrace{-v^b n_b n^a}_{\text{orthogonal}} + \underbrace{(v^a + v^b n_b n^a)}_{\text{tangent}}$$

$= \gamma^a{}_b v^b$ where $\gamma^a{}_b = \delta^a{}_b + n_b n^a$ (*)

$$\hookrightarrow n_a(\gamma^a{}_b v^b) = n_a v^a + \underbrace{n_a n^a}_{-1} v^b n_b$$

$= 0$ ∴

$M \rightarrow \Sigma$

$v^a \rightarrow \gamma^i{}_b v^b$

$$\Rightarrow \gamma_{ab} = \gamma^i{}_a \gamma^j{}_b \gamma_{ij} \overset{(*)}{=} (\delta^i{}_a + n_a n^i)(\delta^j{}_b + n_b n^j)\gamma_{ij}$$

use $\gamma_{ij} = g_{ij}$, $n_a n^a = -1$

$$= g_{ab} + n_a n_b \quad (**)$$

If $u^a n_a = v^a n_a = 0 \Rightarrow \gamma_{ab} u^a u^b = g_{ab} u^a u^b$

But if $u^a = \lambda n^a$, $\Rightarrow \gamma_{ab} u^a v^b = 0$

Intrinsic Curvature

same $(g_{ab}, M) \rightarrow (\gamma_{ij}, \Sigma)$

$D_i \gamma_{jk} = 0$ (also assume torsion free)

$^{(3)}\Gamma^i{}_{jk}$

$$(D_i D_j - D_j D_i) V^k = {}^{(3)}R^k{}_{eij} V^e$$

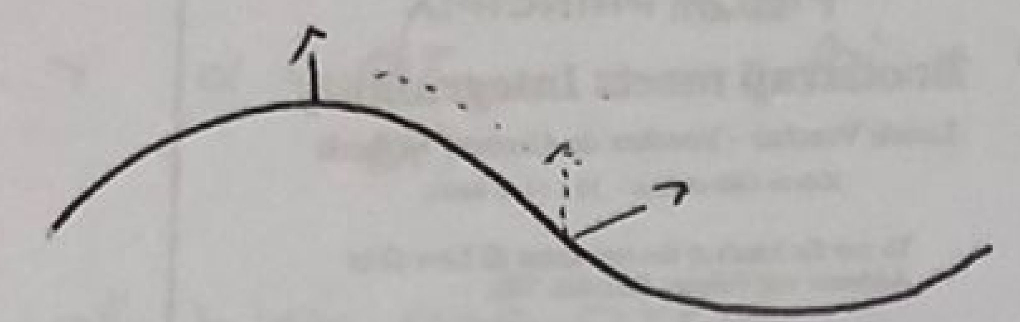
$$^{(3)}R_{ij} = {}^{(3)}R^k{}_{ikj}$$

$$^{(3)}R = \gamma^{ij}\,{}^{(3)}R_{ij}$$

$$D_e T^{b_1 b_2}{}_{c_1 c_2} = \gamma^{b_1}{}_{d_1} \gamma^{b_2}{}_{d_2} \gamma^{e_1}{}_{c_1} \gamma^{e_2}{}_{c_2} \gamma^f{}_e \nabla_f T^{d_1 d_2}{}_{e_1 e_2}$$

$$\underset{(**)}{D_e \gamma_{bc}} = \gamma^d{}_b \gamma^e{}_c \gamma^f{}_e \nabla_f (\underbrace{g_{de}}_{0 \text{ (metric compat)}} + n_d n_e)$$

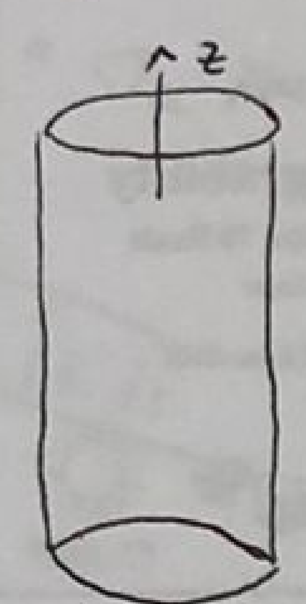
Extrinsic Curvature



Σ

$$K_{ij} = -\nabla_i n_j \; (= \nabla_j n_i) \quad \text{symmetric}$$

example: cylinder as hypersurface on R^3



$$\Sigma = \{ (x, y, z) \in \mathbb{R} \mid \rho^2 = x^2 + y^2 = \overset{R^2}{1} \}$$

$\hookrightarrow t = \rho - 1 = 0$

$$n_e = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0 \right)$$

$$\nabla_b n_e = \begin{pmatrix} \frac{y^2}{\rho^3} & -\frac{xy}{\rho^3} & 0 \\ -\frac{xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\partial n_e}{\partial x^b} \quad \text{(Christoffels vanish)}$$

$$J^e{}_i = \frac{dx^e}{dx^i} : \quad i = \{\phi, z\}, \quad e = \{x, y, z\} \quad \to \tan\phi = \frac{y}{x}, \quad \gamma_{ij} = (R^2, 1)$$

$\hookrightarrow$ induced curv $= 0$

$$\Rightarrow K_{ij} = -\nabla_b n_e J^e{}_i J^b{}_j$$

$$\Rightarrow K_{\phi\phi} = -\frac{(x^2+y^2)^2}{\rho^3}\Big|_{\rho^2 = 1} = -1$$

Foliation

$$M = \{\Sigma_t\}_{t \in \mathbb{R}}, \quad \Sigma_t = \{ x^a \in M, \hat{t}(a) = t \} \text{ slice or leaf}$$

$\hookrightarrow \hat{t}$ smooth

$\nabla_c \hat{t} \neq 0$

$\to M$ is globally hyperbolic

$n_a = -\alpha \nabla_a t \quad , \quad n_a n^a = -1$

$= -(\alpha, \vec{0}) \quad , \quad \alpha = \frac{d\tau}{dt}$

lapse

$g^{ab} n_a n_b = -1 \Rightarrow \alpha = \sqrt{\frac{-1}{g^{tt}}}$

$n^a = \frac{1}{\alpha}(1, -\beta^i) \quad , \quad \beta^i = \alpha^2 g^{ti}$

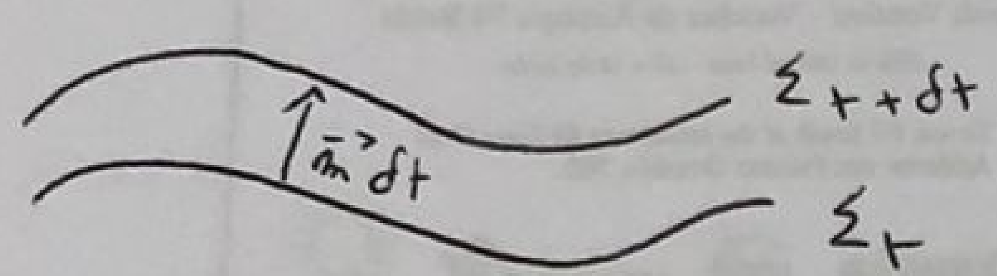
shift

$\to \quad \alpha = \frac{d\tau}{dt} \quad , \quad -\beta^i = \frac{dx^i}{dt}$

Evolution generator

$m_a = \alpha n_a$

$m^a \nabla_a t = 1$



$t(p + \delta t\, \vec{m}) = t(p) + \delta t$

$\mathcal{L}_n \gamma_{ab} = n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_a n^c + \gamma_{ac} \nabla_b n^c$

$\nabla_c \gamma_{ab} = \nabla_c (n_a n_b)$

$= -2K_{ab}$

$\mathcal{L}_m \gamma_{ab} = -2\alpha K_{ab}$

${}^{(3)}R_{abc}{}^{d} u_d = (D_a D_b - D_b D_a) u_c$

$D_a D_b u_c = \gamma^d_a \gamma^e_b \gamma^f_c \nabla_d (\gamma^{eh}_e \gamma^g_f \nabla_g u_h)$

$\vdots$

${}^{(3)}R^c{}_{dab} = \gamma^e_a \gamma^f_b \gamma^c_h \gamma^g_d R^h{}_{gef} - K^c{}_a K_{db} + K^c{}_b K_{ad}$

${}^{(3)}R_{db} + K K_{db} - K^a{}_b K_{ad} = \gamma^f_b \gamma^a_h \gamma^g_d R^h{}_{gaf} \quad (\Diamond)$

${}^{(3)}R + K^2 - K_{ab} K^{ab} = R + 2 R_{ab} n^a n^b \quad (\square)$

$$R^c{}_{dab}\, n^d = (\nabla_a \nabla_b - \nabla_b \nabla_a)\, n^c$$

$$\gamma^c{}_d\, n^e\, \gamma^g{}_a\, \gamma^g{}_b\, R^d{}_{egg} = D_b K^c{}_a - D_a K^c{}_b$$

$$n^e\, \gamma^g{}_b\, R_{eg} = D_b K - D_a K^a{}_b$$

. . . .

$\gamma^4 R \quad , \quad \gamma^3 n R \quad , \quad \gamma^2 n^2 R \quad , \quad n^3 R = 0$ Gauss-Codazzi relations

$$\gamma_{ac}\, n^d\, \gamma^e{}_b\, R^c{}_{ged}\, n^g = \gamma_{ac}\, n^d\, \gamma^e{}_b\, (\nabla_e \nabla_d n^c - \nabla_d \nabla_e n^c)$$

$$[\nabla_b n_e = -K_{eb} - D_e \ln\alpha\, n_b]$$

$$= -K_{eg} K^g{}_b + \frac{1}{\alpha} D_b D_e \alpha + \gamma^c{}_a\, \gamma^e{}_b\, n^g \nabla_g K_{ce}$$

$$\gamma_{ac}\, n^e\, \gamma^e{}_b\, R^c{}_{ged}\, n^g = \frac{1}{\alpha} \mathcal{L}_n K_{ab} + \frac{1}{\alpha} D_a D_b \alpha + K_{ag} K^g{}_b \quad (\Delta)$$

. . . . .

$$R_{ab} - \frac{1}{2} R\, g_{ab} = 8\pi\, T_{ab}$$

$\rightarrow E = T_{ab}\, n^a n^b$ energy density

$p^a = -\gamma^a{}_c\, n^b\, T_{cb}$ momentum density

$S_{ab} = \gamma^c{}_a\, \gamma^d{}_b\, T_{cd}$ matter stress tensor

$\rightarrow T = g^{ab}\, T_{ab} = S - E$

e.g. Fluid: $T_{ab} = (e + p)\, u_a u_b + p\, g_{ab}$

$\rightarrow E = (e + p)\, W^2 - p$

$p^a = (E + p)\, U^a$

$S_{ab} = (E + p)\, U_a U_b + p\, \gamma_{ab}$

where $W = -n_a u^a$ Lorentz factor

$U^a = \frac{1}{W} \gamma^a{}_b\, u^b$ Eulerian velocity

Einstein equations

$$\rightarrow R_{ab} n^a n^b + \frac{1}{2} R = 8\pi E$$

$\downarrow$ ($\square$)

$$^{(3)}R + K^2 - K_{ab}K^{ab} = 16\pi E \qquad \text{Hamiltonian constraint}$$

$$\rightarrow R_{ab}\gamma^a{}_c n^b \underbrace{- \frac{1}{2} R n_a \gamma^a{}_c}_{0} = -8\pi p_c$$

$\downarrow$ (☺)

$$D_a K^a{}_b - D_b K = 8\pi p_b \qquad \text{Momentum constraint}$$

$\rightarrow$ trace-reversed EE: $R_{ab} = 8\pi \left(T_{ab} - \frac{1}{2} T g_{ab}\right)$

$$\gamma^a{}_c \gamma^b{}_d R_{ab} = 8\pi \left[S_{cd} - \frac{1}{2}(S - E)\gamma_{cd}\right]$$

$\downarrow$ ($\triangle$, $\diamond$)

$$-\frac{1}{2}\mathcal{L}_n K_{cd} - \frac{1}{\alpha} D_c D_d \alpha + {}^{(3)}R_{cd} + K K_{cd} - 2K_{ca}K^a{}_d = 8\pi [\,..\,]$$

$$\rightarrow \mathcal{L}_n \gamma_{ab} = -2\alpha K_{ab}$$

.....

$$E^{ab} = G^{ab} - 8\pi T^{ab}$$

constraints: $E^{ab} n_b = 0$, $E^{tb} = 0$

$$\rightarrow \nabla_a E^{ab} = \underbrace{\nabla_a G^{ab}}_{\overset{\text{Bianchi}}{=} 0} - 8\pi \underbrace{\nabla_a T^{ab}}_{\overset{\text{conservation}}{=} 0} = 0$$

$$\nabla_t E^{tb} = \text{terms involving } E^{ab},\ \partial_i E^{ab}$$

Maxwell equations

$\nabla \cdot \vec{E} = \rho$, $\nabla \cdot \vec{B}$ constraints

$\partial_t \vec{E} = \nabla \times \vec{B} - \vec{J}$; $\partial_t \vec{B} = -\nabla \times \vec{E}$ Evolution eqns

$\rightarrow \partial_t (\nabla \cdot \vec{B}) = 0$

$\partial_t (\nabla \cdot \vec{E}) = -\nabla \cdot \vec{J}$

but charge conservation $\partial_t \rho + \nabla \cdot \vec{J} = 0$

$\rightarrow \partial_t (\nabla \cdot \vec{E} - \rho) = 0$