Homework sheet 7

TAs: Marcus Haberland & Aldo Gamboa

$$S = \frac{1}{32\pi G}\int dt\, d^3x \left(4\phi\Delta\phi - \frac{1}{c^2} A_i \Delta A_i\right)$$

$$+ \sum_{a=1,2} \int dt \left[-m_a \phi + \underline{\frac{1}{c^2}\left(\frac{3}{2} v_a^i S_a^{ij} \partial_j \phi + m_a v_a^i A_i + \frac{1}{2} S^{ij} \partial_i A_j\right)}\right]$$

$$\underline{\underline{- \frac{1}{8\pi G c^2} \int dt\, d^3x\, \phi\, \partial_t^2 \phi}}$$

$$\underline{\underline{\underline{+ \sum_{a=1,2} \int dt\, m_a \left[-c^2 + \frac{v_a^2}{2} + \frac{1}{c^2}\left(\frac{v_a^4}{8} - \frac{3v_a^2}{2}\phi - \frac{1}{2}\phi^2\right)\right]}}} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

a) For the leading order spin-orbit part, we only need the non-spinning part of J_ϕ, J_A^i, D_ϕ^{-1} and D_A^{-1} to their respective lowest order c^{-0}.
Note that S^{ij} terms are already suppressed by $\frac{1}{c^2}$, then we can drop the $\frac{1}{c^2}$ suppressed ϕ-term and the kinetic Lagrangian part, which doesn't depend on the fields and therefore couples to them only at higher orders.

b) The spin terms couple linearly to ϕ, A_i

$\rightsquigarrow$ Modify J_ϕ & J_A^i

$$J_\phi = -m_1 \delta_1 - \frac{3}{2c^2} v_1^i S_1^{ij} \partial_j \delta_1 + \dots + (1 \leftrightarrow 2)$$

$$J_A^i = m_1 v_1^i \delta_1 - \frac{1}{2} S^{ji} \partial_j \delta_1 + \dots + (1 \leftrightarrow 2)$$

$$D_\phi = -\frac{1}{4\pi G}\Delta \;; \qquad D_A = \frac{1}{16\pi G}\Delta$$

This can be seen from the variation of the action with respect to e.g. ϕ:

$$\delta S = \int dt\, d^3x \left(\frac{4}{32\pi G} \delta\phi \Delta\phi + \frac{4}{32\pi G} \phi \Delta \delta\phi \right.$$

2x partial integration

$$\left. \sum_{a=1,2} \left[- m_a \delta_a \delta\phi + \frac{1}{c^2} \frac{3}{2} v_a^i S_a^{ij} \delta_a \partial_j \delta\phi \right] \right) + \mathcal{O}(c^{-2})$$

1x partial integration

$$= \int dt\, d^3x\, \delta\phi \left(\underbrace{\frac{1}{4\pi G} \Delta\phi}_{\equiv -D_\phi \phi} + \underbrace{\sum_{a=1,2} \left[- m_a \delta_a - \frac{3}{2c^2} v_a^i S_a^{ij} \partial_j \delta_a \right]}_{\equiv J_\phi} \right) + \mathcal{O}(c^{-2})$$

and similarly for J_A^i, where we note that $S^{ij} = -S^{ji}$ as S is antisymmetric and therefore $S_a^{ij} \partial_i A_j = -S^{ji} \partial_i A_j = -S^{ij} \partial_j A_i$ after re-labeling.

The action then remains (see lectures)

$$S = \int dt\, \mathcal{L}_{kin} + \int dt\, d^3x \left[J_\phi \phi - \frac{1}{2} \phi D_\phi \phi + \frac{1}{c^2} J_A^i A_i - \frac{1}{2c^2} A_i D_A A_i \right] + \mathcal{O}(c^4)$$

with associated field equations

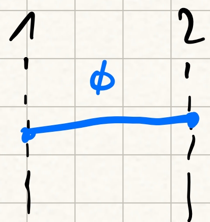
$$D_\phi \phi = J_\phi \quad \& \quad D_A A^i = J_A^i$$

$$\Rightarrow \quad \phi = D_\phi^{-1} J_\phi \quad \& \quad A^i = D_A^{-1} J_A^i$$

c) Let's insert the solution into the action to get to the Fokker action:

$$S = \int dt\, \mathcal{L}_{kin} + \int dt\, d^3x \left[\frac{1}{2} J_\phi \mathcal{D}_\phi^{-1} J_\phi + \frac{1}{2c^2} J_A^i \mathcal{D}_A J_A^i \right]$$

Computing the Fokker action:

S_1 

$$\equiv \frac{1}{2} \int d^3x\, J_\phi \mathcal{D}_\phi^{-1} J_\phi$$

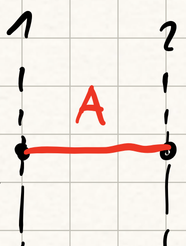
2 identical terms

$$= \frac{2}{2} \int d^3x \left(-\frac{3}{2c^2} v_1^i S_1^{ij} \partial_j \delta_1 \right) \left(-4\pi G \Delta^{-1} \right) \left(-m_2 \delta_2 \right)$$

$$= \int d^3x \left(-\frac{3}{2} \frac{G m_2}{c^2} v_1^j S^{ij} \partial_j \frac{1}{r} \right)$$

$$= \int d^3x \underbrace{\frac{1}{c^2} \frac{3}{2} \frac{G m_2}{r^2} \left(v_1^j S_1^{ij} n^j \right)}_{\mathcal{L}_{SO}^\phi}$$

where we have used $\Delta^{-1} \delta_a = -\frac{1}{4\pi |\vec{x} - \vec{x}_a|}$

S_1 

$$\equiv \frac{1}{2c^2} \int d^3x\, J_A^i \mathcal{D}_A^{-1} J_A^i$$

2 identical terms

$$= \frac{2}{2c^2} \int d^3x \left(\frac{1}{2} S_1^{ij} \partial_j \delta_1 \right) \left(16\pi G \Delta^{-1} \right) \left(m_2 v_2^i \delta_2 \right)$$

$$= \int d^3x \; \frac{2Gm_2}{c^2} v_2^i S_1^{ij} \partial_j \frac{1}{r}$$

$$= \int d^3x \; \underbrace{\frac{1}{c^2}\left(-\frac{2Gm_2}{r^2} v_2^i S_1^{ij} n^j\right)}_{\mathcal{L}_{SO}^{A}}$$

$$\Rightarrow \mathcal{L}_{SO} = \mathcal{L}_{SO}^{\phi} + \mathcal{L}_{SO}^{A}$$

$$= \frac{1}{c^2} \frac{Gm_2}{2r^2} S_1^{ij} n^j (3v_1^i - 4v_2^i) + (1 \leftrightarrow 2)$$

d) To compute the Hamiltonian, we use the leading order relations

$$v_a^i = \frac{p_a^i}{m_a} + \mathcal{O}\left(\frac{1}{c^2}\right)$$ and flip the sign of $\mathcal{L}_{SO}$ (see the solution of HW3, Ex II):

$$\Rightarrow H_{SO} = -\frac{1}{c^2} \frac{G}{r^2} S_1^{ij} n^j \left(\frac{3}{2}\frac{m_2}{m_1} p_1^i - 2p_2^i\right) + (1 \leftrightarrow 2)$$

Going to center of mass coordinates $\vec{p} = \vec{p}_2 = -\vec{p}_1$

$$H_{SO} = -\frac{1}{c^2} \frac{G}{r^2} S_1^{ij} p^i n^j \left(\frac{3}{2}\frac{m_2}{m_1} + 2\right) + (1 \leftrightarrow 2)$$

And the angular momentum becomes

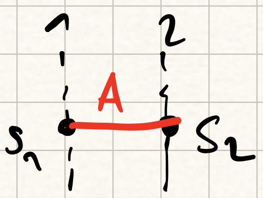
$$L^i = \varepsilon^{ijk}(x_1^j p_1^k + x_2^j p_2^k) = -r\,\varepsilon^{ijk} n^j p^k$$

in the center of mass

$$\Rightarrow \quad H_{SO} = \frac{1}{c^2}\,\frac{G}{r^3}\left(\frac{3}{2}\frac{m_2}{m_1}+2\right)(\vec{L}\cdot\vec{S}_1) + (1 \leftrightarrow 2)$$

where $S_a^i = \varepsilon^{ijk} S_a^{jk}$ $\rightsquigarrow$ similar to atomic physics!

e) $S_1 - S_2$ contribution:



$$\equiv \frac{1}{2c^2}\int d^3x\, J_A^i\, D_A^{-1}\, J_A^i$$

$$= \frac{2}{2c^2}\int d^3x \left(S^{ij}\partial_j \delta_1\right)\left(16\pi G\,\Delta^{-1}\right)\left(\frac{1}{2}S^{ik}\partial_k\delta_2\right)$$

$$= \int d^3x\, \frac{1}{c^2}\, G\, S_1^{ij} S_2^{ik}\, \partial_j\partial_k\left(\frac{1}{r}\right)$$

$$= \int d^3x\, \frac{1}{c^2}\, G\left(S_1^i S_2^i\,\delta^{jk} - S_1^j S_2^k\right)\frac{1}{r^3}\left(-\delta^{jk}+3n^j n^k\right)$$

$$= \int d^3x\, \frac{1}{c^2}\, G\left[(\vec{S}_1\cdot\vec{S}_2) - 3(\vec{S}_1\cdot\vec{n})(S_2\cdot\vec{n})\right]$$

where we have used the identities

$$S_1^{ij} S_2^{jk} = S_1^i S_2^i\,\delta_{jk} - S_1^k S_2^j \quad \&$$

$$\partial_j\partial_k \frac{1}{r} = -\partial_j\left(\frac{n^k}{r^2}\right) = \frac{1}{r^3}\left(\underbrace{-\delta_{jk}+3n^j n^k}_{\text{traceless!}}\right)$$

And to obtain the Hamiltonian, flip the sign:

$$H_{S_1S_2} = -\left(\frac{1}{c^2}\right)\frac{G}{r^3}\left[(\vec{S}_1\cdot\vec{S}_2) - 3(\vec{S}_1\cdot\vec{n})(S_2\cdot\vec{n})\right]$$