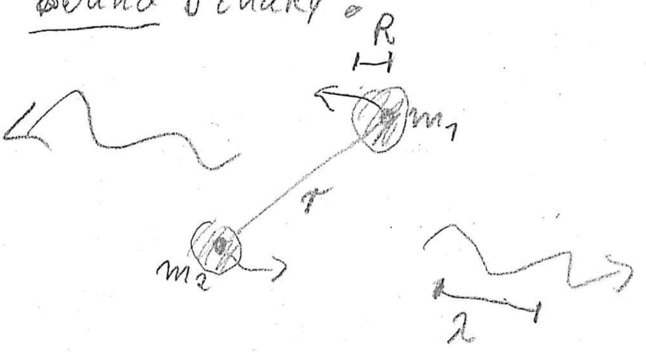
post-Newtonian (PN) approximation

bound binary:



if $R < r < \lambda$ (hierarchy of scales)

then PN approx. possible.

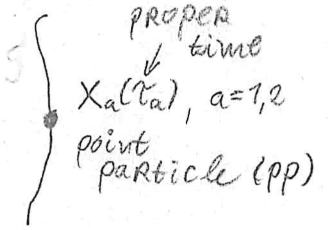
expansion parameter:

$$\boxed{\epsilon \doteq \frac{r^2}{\lambda^2} \sim \frac{v^2}{c^2} \sim \frac{G(m_1+m_2)}{c^2 \cdot r}}$$

$\uparrow$ $\lambda \sim \frac{cr}{v}$ $\quad$ $\uparrow$ 3rd Kepler (bound orbits)

$\hookrightarrow$ $\boxed{\text{one PN order} \;\hat{=}\; \frac{1}{c^2}}$

compact object ⇝ multipole expansion in $\frac{R}{r}$



action:

$$S_a^{pp} = -\int d\tau_a \, \underbrace{m_a c^2}_{\text{point mass}} + \mathcal{O}\left(\frac{R^2}{r^2}\right)$$

more later...

$\hookrightarrow$ spin ⇝ exercise (∂g-coupling)

PN approx. for orbital motion:

the Fokker action

[Remark: mistakes in literature]

- starting point: $S = \underset{\substack{\uparrow \\ \text{Einstein action}}}{S_E} + \overset{\text{body 1}}{S_1^{pp}} + \overset{\text{body 2}}{S_2^{pp}}$
- field eq. for $g_{\mu\nu}$ from S: $\delta S = \underbrace{\frac{\delta S}{\delta g_{\mu\nu}}}_{\overset{!}{=} 0 \quad (*)} \delta g_{\mu\nu}$
- solve perturbatively in $\epsilon \sim \frac{1}{c^2}$

  $\hookrightarrow g_{\mu\nu}[X_1, X_2]$
- insert solution into $S \longrightarrow S'[X_a^\mu] = S[g_{\mu\nu}[X_a], X_a^\mu]$ (Fokker action)

then:

$$\delta S' = \Bigg(\underbrace{\frac{\delta S}{\delta g_{\mu\nu}}\bigg|_{g = g[X_a]}}_{0 \text{ since } g_{\mu\nu}[X_a] \text{ satisfies } (*)} \cdot \frac{\delta g_{\mu\nu}[X_a]}{\delta X_a^\rho} + \frac{\delta S}{\delta X_a^\rho} \Bigg) \delta X_a^\rho$$

$$\rightsquigarrow \frac{\delta S'}{\delta X_a} = \frac{\delta S}{\delta X_a} \overset{!}{=} 0 \quad \rightsquigarrow \text{EOM for } X_a \text{ unchanged!}$$

Let's calculate to 1PN:

ansatz for $g_{\mu\nu}$:

$$c^2 d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$$
$$\overset{!}{=} e^{2\phi/c^2}\left(c\cdot dt - \frac{1}{c^2}A_i dx^i\right)^2 - e^{-2\phi/c^2}\left(\delta_{ij} + \frac{1}{c^4}\sigma_{ij}\right)dx^i dx^j$$

"PN fields" ϕ, A_i, σ_{ij}

gravito-electric (ϕ) gravito-magnetic (A_i)

point mass:

$$S_1^{pp} = -m_1 c^2 \int dt \frac{d\tau}{dt} + \mathcal{O}\left(\frac{R^2}{r^2}\right) \qquad \boxed{v_1^i = \frac{dx_1^i}{dt}}$$
$$= \int dt\, m_1\left[-c^2 + \frac{1}{2}v_1^2 - \phi + \frac{1}{c^2}\left(\frac{1}{8}v_1^4 - \frac{3}{2}v_1^2\phi - \frac{1}{2}\phi^2 + A_i v_1^i\right) + \mathcal{O}(c^{-4})\right]_{\vec{x}=\vec{x}_a}$$

(exercise: include spin ⇝ gravito-magnetic dipole)

Einstein action:

$$S_E = \frac{c^4}{16\pi G}\int dt\, d^3x \sqrt{-g}\, R$$
$$= \frac{1}{32\pi G}\int dt\, d^3x\left(4\phi \overset{\partial_i\partial_i}{\Delta}\phi - \frac{4}{c^2}\phi\partial_t^2\phi - \frac{1}{c^2}A_i\Delta A_i + \mathcal{O}(c^{-4})\right) + \text{boundary terms}$$

in Lorenz gauge $\partial_\nu \bar{h}^{\mu\nu} = 0 \Leftrightarrow \partial_i A_i + 4\partial_t\phi = 0$

σ_{ij} is higher order

sort terms:

$$S = \int dt\, L_{kin} + \int dt\, d^3x\left(j_\phi\phi - \frac{1}{2}\phi\mathcal{D}_\phi\phi + \frac{1}{c^2}j_A^i A_i - \frac{1}{2c^2}A_i\mathcal{D}_A A_i\right) + \mathcal{O}(c^{-4})$$

$$L_{kin} = m_1\left(-c^2 + \frac{v_1^2}{2} + \frac{v_1^4}{8c^2} + \dots\right) + (1 \leftrightarrow 2)$$

$$j_\phi = -m_1\left(1 + \frac{3v_1^2}{2c^2}\right)\delta_1 + (1 \leftrightarrow 2) \qquad \text{where } \boxed{\delta_1 \equiv \delta(x - x_1(t))}$$

$\hookrightarrow 1 = \int d^3x\, \delta_1$

$$\mathcal{D}_\phi = -\frac{1}{4\pi G}\left(\Delta - c^{-2}\partial_t^2\right) + \frac{m_1}{c^2}\delta_1 + \frac{m_2}{c^2}\delta_2$$

$$j_A^i = m_1 v_1^i \delta_1 + (1 \leftrightarrow 2), \quad \mathcal{D}_A = \frac{1}{16\pi G}\Delta$$

field eqs.: (vary ϕ & A_i)

$$\mathcal{D}_\phi\phi = j_\phi, \quad \mathcal{D}_A A^i = j_A^i \qquad \text{(exercise)}$$

solutions:

$$\phi = \mathcal{D}_\phi^{-1} j_\phi, \quad A^i = \mathcal{D}_A^{-1} j_A^i$$

insert into $S \rightarrow S'$:

$$S' = \int dt\, L_{kin} + \int dt\, d^3x \Big(\underbrace{\mathcal{J}_\phi \mathcal{D}_\phi^{-1} \mathcal{J}_\phi - \tfrac{1}{2} (\mathcal{D}_\phi^{-1} \mathcal{J}_\phi) \overbrace{\mathcal{D}_\phi \mathcal{D}_\phi^{-1}}^{1} \phi}_{\frac{1}{2} \mathcal{J}_\phi \mathcal{D}_\phi^{-1} \mathcal{J}_\phi} + \tfrac{1}{2c^2} \mathcal{J}_A^i \mathcal{D}_A^{-1} \mathcal{J}_A^i \Big) + \mathcal{O}(c^{-4})$$

evaluate A-term: (the last one)

with formula: $\boxed{\Delta^{-1} \delta_a = - \frac{1}{4\pi |\vec{x} - \vec{x}_a|}}$

$$L_A = \int d^3x \frac{1}{2c^2} \mathcal{J}_A^i \mathcal{D}_A^{-1} \mathcal{J}_A^i = \int \frac{d^3x}{2c^2} (m_1 V_1^i \delta_1 + m_2 V_2^i) \left(\frac{\Delta}{16\pi G} \right)^{-1} (m_1 V_1^i \delta_1 + m_2 V_2^i \delta_2)$$

$$= \int d^3x \frac{8\pi G}{c^2} \Big(\underbrace{m_1 V_1^i \delta_1 \Delta^{-1} m_1 V_1^i \delta_1}_{\sim \frac{1}{|\vec{x}_1 - \vec{x}_1|}} + \underbrace{m_1 V_1^i \delta_1 \Delta^{-1} m_2 V_2^i \delta_2 + m_2 V_2^i \delta_2 \Delta^{-1} m_1 V_1^i \delta_1}_{2 m_1 V_1^i \delta_1 \Delta^{-1} m_2 V_2^i \delta_2} + \underbrace{m_2 V_2^i \delta_2 \Delta^{-1} m_2 V_2^i \delta}_{\sim \frac{1}{|\vec{x}_2 - \vec{x}_2|}}$$

part. int.

divergent self-energy
↳ drop

↳ drop

$$= - \frac{4G}{c^2 r} m_1 m_2 V_1^i V_2^i$$

where $r = |\vec{x}_1 - \vec{x}_2|$

ϕ-term:

$$L_\phi = \int d^3x \frac{1}{2} \mathcal{J}_\phi \mathcal{D}_\phi^{-1} \mathcal{J}_\phi$$

$2 \times 4 \times 2 = 16$ terms
↳ use diagrams to keep track

first: operator inverse

$$A = \Delta + \varepsilon B \rightsquigarrow A^{-1} = \Delta^{-1} - \varepsilon \Delta^{-1} B \Delta^{-1} + \mathcal{O}(\varepsilon^2)$$

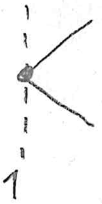
$$\text{check: } AA^{-1} = 1 + \varepsilon (B\Delta^{-1} - \underbrace{\Delta\Delta^{-1}}_{1} B \Delta^{-1}) + \mathcal{O}(\varepsilon^2) \overset{!}{=} 1 + \mathcal{O}(\varepsilon^2)$$

$$\mathcal{D}_\phi^{-1} = \left(- \frac{\Delta}{4\pi G} + \frac{\partial_t^2}{4\pi G c^2} + \frac{m_1}{c^2} \delta_1 + \frac{m_2}{c^2} \delta_2 \right)^{-1}$$

$$= \underbrace{-4\pi G \Delta^{-1}}_{} \underbrace{- \frac{4\pi G}{c^2} \Delta^{-1} \partial_t^2 \Delta^{-1}}_{} \underbrace{- \frac{(4\pi G)^2}{c^2} m_1 \Delta^{-1} \delta_1 \Delta^{-1}}_{} \underbrace{- \frac{(4\pi G)^2}{c^2} m_2 \Delta^{-1} \delta_2 \Delta^{-1}}_{} + \mathcal{O}(c^{-4})$$


①


②



③

④

$$\mathcal{J}_\phi = \underbrace{-m_1\left(1+\frac{3v_1^2}{2c^2}\right)\delta_1}_{\mathcal{J}_{\phi 1}}\underbrace{-m_2\left(1+\frac{3v_2^2}{2c^2}\right)\delta_2}_{\mathcal{J}_{\phi 2}}$$

contributions from ①:

$$= \frac{1}{2}\int d^3x\left[\mathcal{J}_{\phi 1}(-4\pi G\Delta^{-1})\mathcal{J}_{\phi 2} + \mathcal{J}_{\phi 2}(-4\pi G\Delta^{-1})\mathcal{J}_{\phi 1}\right]$$

part. int.

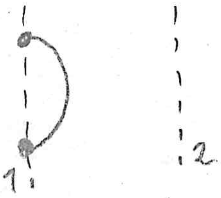
(2 terms)

2 id. terms

$$= \frac{2}{2}\cdot\int d^3x\,(-4\pi G)\,\mathcal{J}_{\phi 1}\Delta^{-1}\mathcal{J}_{\phi 2}$$

$$= \ldots = \frac{Gm_1m_2}{r}\left(1+\frac{3v_1^2}{2c^2}+\frac{3v_2^2}{2c^2}\right)+O(c^{-4})$$

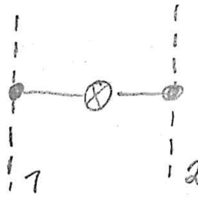
divergent self-int. ⇝ drop



&

(2 terms)

from ②: 2 id. terms



$$= \frac{2}{2}\int d^3x\,\mathcal{J}_{\phi 1}\frac{(-4\pi G)}{c^2}\Delta^{-1}\partial_t^2\Delta^{-1}\mathcal{J}_{\phi 2}$$

(2 terms)

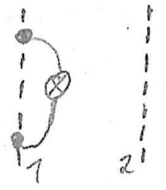
$$= \int d^3x\,\frac{4\pi G}{c^2}\underbrace{(\partial_t\mathcal{J}_{\phi 1})}_{\approx -m_1\delta_1}(\partial_t\Delta^{-2}\underbrace{\mathcal{J}_{\phi 2}}_{\approx -m_2\delta_2}) + \underbrace{(\text{total time der.})}_{\text{drop}}$$

$$= \ldots = \frac{G}{2rc^2}\left(v_1^iv_2^i - v_1^in^iv_2^jn^j\right)+O(c^{-4})$$

$$\Delta^{-2}\delta_2 = -\frac{1}{8\pi}|\vec{x}-\vec{x}_2(t)|$$

$$n^i = \frac{x_1^i - x_2^i}{r}$$

drop:



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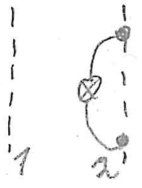
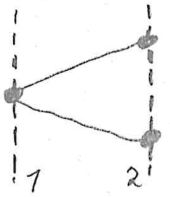


(2 terms)

from ③:



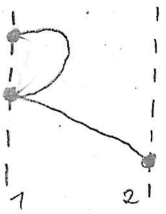
part. int.

$$= \underbrace{\frac{1}{2}}_{\text{"symmetry factor"}} \int d^3x \, \underbrace{J_{\phi 2}}_{\approx -m_2\delta_2} \left(-\frac{(4\pi G)^2}{c^2} m_1 \Delta^{-1} \delta_1 \Delta^{-1} \right) \underbrace{J_{\phi 2}}_{\approx -m_2\delta_2}$$

(1 term)

$$= -\frac{G^2 m_2^2 m_1}{2c^2 r^2}$$

drop:



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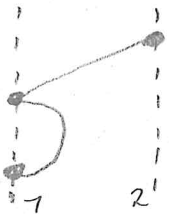

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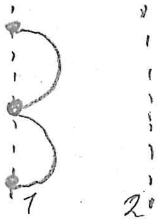
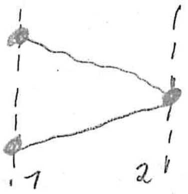


(3 terms)

from ④:



$$= -\frac{G^2 m_1^2 m_2}{2c^2 r^2}$$

analogous to ③
& drop 3 terms

(1+3 terms)

(total: 16 terms)

add: $L_{kin} + L_A + L_\phi$ = sum of diagrams

↳ eq.(2) on exercise sheet 3

connection to quantum field theory (QFT)
and effective field theory

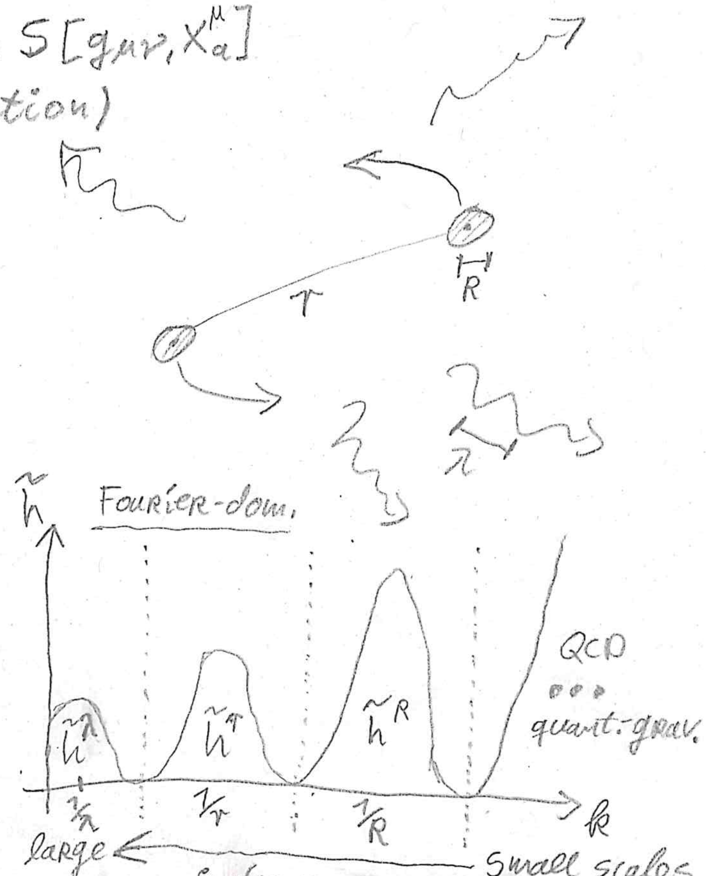
QFT: allow for all possible diagrems

e.g.  ⤳ quantum-grav. correction

derivation: eg. via path integral

↳ "ingtegrate out" $g_{\mu\nu}$ from $S[g_{\mu\nu}, x^\mu_a]$

(classical part: Fokker action)



EFT:

split: $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$

$= h^\lambda_{\mu\nu} + h^r_{\mu\nu} + h^R_{\mu\nu} + \dots$

according to hierarchy of scales

$\lambda > r > R > \dots$

↳ decoupling of Fourier modes:

$\tilde{h}^\lambda \cdot \tilde{h}^r = 0$

$\tilde{h}^\lambda \cdot \tilde{h}^R = 0$

$\tilde{h}^r \cdot \tilde{h}^R = 0$

...

EFT approach:

successively integrate out (path integral over)
small scales ⤳ coarse-graining, averaging
order: $h^R_{\mu\nu} \to h^r_{\mu\nu} \to h^\lambda_{\mu\nu}$ ⤳ by Fokker action

integrate out $h^R_{\mu\nu}$:

↳ explicit calculation impossible due to strong gravity

↳ make ansatz based on

symmetries & power counting
(general cov.) ($\frac{R}{r}$, multipole expansion)

$$S^{pp} = -\int d\tau \Big(\underbrace{mc^2}_{\text{const. mass}} + \frac{1}{2} \underbrace{\Omega^{\mu\nu}}_{\text{angular vel.}} \underbrace{S_{\mu\nu}}_{\text{spin}} + \frac{1}{2} \underbrace{E_{\mu\nu}}_{\text{electric curvature component}} \underbrace{Q^{\mu\nu}_E}_{\text{gravito-electric quadrupole}} + \frac{1}{2} \underbrace{B_{\mu\nu}}_{\text{magnetic curvature comp.}} \underbrace{Q^{\mu\nu}_B}_{\text{gravito-magnetic quadrupole}} + O\Big(\frac{R^4}{r^4}\Big)\Big)$$

+ need model for multipoles

e.g. adiabatic tides: $Q^{\mu\nu}_E = \text{const} \cdot E^{\mu\nu}$

↳ match to numerics