Black hole perturbation theory and quasinormal modes. 25,26 / 11 / 24.

## General references :


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- K. D. Kokkotas & B. G. Schmidt, "Quasinormal modes of stars and black holes." gr-qc / 9909058
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- A. Pound & B. Wardell, "Black hole perturbation theory and gravitational self-force." 2101.04592
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# LECTURE 1.

Start by studying a test scalar field propagating on a black-hole spacetime (SCHWARZSCHILD).

$$\Box \Phi = 0 \tag{1}$$

$$ds^2 = - f(r)\, dt^2 + f(r)^{-1} dr^2 + r^2 \left( d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \right) \tag{2}$$

$$f(r) := 1 - 2M/r = 1 - r_H/r \qquad (r_H = 2M). \tag{3}$$

$c = G = 1.$

In general

$$\Box \Phi = \nabla_\mu \nabla^\mu \Phi \overset{\text{EXERCISE}}{=} \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi \right) = 0 \tag{4}$$

$g := \det(g_{\mu\nu})$

$= r^2 \sin\vartheta$

Try to separate variables, assuming harmonic time-dependence:

$$\Phi(t, r, \vartheta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{r} \phi(r) \underbrace{Y_{\ell m}(\vartheta, \varphi)}_{\text{spherical harmonics}} e^{-i\omega t} \tag{5}$$

Substitute (5) in (4). Separate variables using: $\underbrace{P_\ell(\vartheta)}_{\text{LEGENDRE polynomials.}} e^{+im\varphi}$ $\quad \ell = 0, 1, \dots$ $\quad |m| \leq \ell$

$$\frac{1}{\sin\vartheta} \frac{d}{d\vartheta} \left[ \sin\vartheta \frac{d}{d\vartheta} P_\ell(\vartheta) \right] - \frac{m^2}{\sin^2\vartheta} P_\ell(\vartheta) = - \ell(\ell+1) P_\ell(\vartheta)$$

and find:

$$\left(1 - \frac{2M}{r}\right)^2 \frac{d^2\phi}{dr^2} + \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right) \frac{d\phi}{dr} + \left\{ \omega^2 - \left(1 - \frac{2M}{r}\right) \left[ \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} \right] \right\} \phi = 0 \tag{6}$$

We can eliminate the $d\phi/dr$ term by a smart choice of radial coordinate

$$\frac{dr}{dx} = 1 - \frac{2M}{r} \quad \therefore \quad x = r + 2M \log\left( \frac{r}{2M} - 1 \right) \tag{7}$$

<u>Obs.</u>: as $r \to 2M$, $x \to -\infty$

as $r \to +\infty$, $x \to +\infty$

Inverse:

$$r(x) = 2M \left[ 1 + W(z) \right] \; ; \qquad z = \exp\left[ x/(2M) - 1 \right]$$

$\uparrow$ (principal branch of the) LAMBERT W-function.

In terms of x:

$$\boxed{\begin{aligned} &\frac{d^2\psi}{dx^2} + \left[\omega^2 - V_\ell(r)\right]\psi = 0 \qquad (8) \\ &V_\ell(r) = \left(1 - \frac{2M}{r}\right)\cdot\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right] \end{aligned}}$$

(effective potential) (9)

That is, we reduced the interaction of $\Phi$ with SCHWARZSCHILD to a one-dimensional equation, where the field interacts with a potential $V_\ell(r)$. No $m$ (why?).

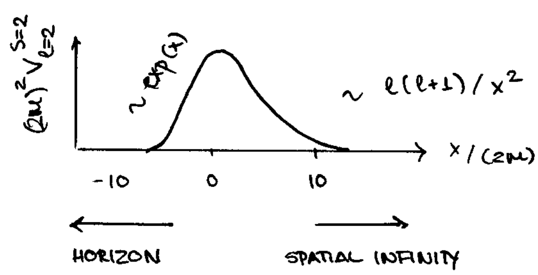
EXERCISE: ~ fill the steps leading to Eq. (8).

~ produce plots of $V_\ell(r)$ for different $\ell$.

More generally, for massless bosonic fields, we have:

$$V_{s\ell}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right] \qquad (10)$$

$$s = \begin{cases} 0 & (\text{SCALAR}) \quad \ell \geq 0 \\ 1 & (\text{VECTOR}) \quad \ell \geq 1 \\ 2 & (\text{TENSOR}) \quad \ell \geq 2 \end{cases}$$




s = 2 : "REGGE-WHEELER" EQUATION (1957). (derivation tomorrow)

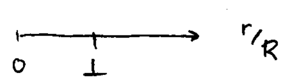
1970: VISHVESHWARA scattered s = 2, l = 2 waves. Saw characteristic, "universal" ringdown

1971: PRESS identifies ringdown with "free oscillation" modes of a black hole working in the $\ell \to \infty$ limit. "Loosely speaking, the black hole vibrates around spherical symmetry in a quasi-normal mode, and the mode is slowly damped by gravitational radiation."

PAUSE TO SHOW VIDEOS.

NB.: For stars, eq. (10) is also valid in vacuum (axial perturbations). Stars also have QNMs.

Cf.: THORNE - CAMPOLATTARO AND CHANDRASEKHAR - FERRARI



What determines the free oscillations? Boundary conditions.

Note: as $x \to \pm\infty$ (i.e., $r \to \begin{cases} +\infty \\ 2M \end{cases}$), Eq. (8) becomes:

$$\frac{d^2\psi}{dx^2} + \omega^2 \psi = 0$$

then:

$$\psi \sim e^{\pm i\omega x} \quad \text{as} \quad x \to \pm\infty$$

Recall, our FOURIER convention:

$$\psi \sim e^{-i\omega t} \, e^{\pm i\omega x} \quad \text{as} \quad x \to \pm\infty$$

Now:

as $x \to -\infty$, we don't want waves coming <u>out</u> of the horizon

$$\therefore \quad \psi \sim e^{-i\omega t} \, e^{-ix\omega}$$

as $x \to +\infty$, we don't want waves coming <u>in</u> from infinity (free oscillations)

$$\therefore \quad \psi \sim e^{-i\omega t} \, e^{+ix\omega}$$

This leads to a boundary value problem:

find $\omega$ such that $\dfrac{d^2\psi}{dx^2} + [\omega^2 - V_e(r)]\,\psi = 0$ with $\begin{array}{ll} \psi \sim e^{-i\omega(t+x)} & x \to -\infty \\ \psi \sim e^{-i\omega(t-x)} & x \to +\infty \end{array}$

These $\omega$ are known as quasinormal frequencies; $\psi$ are the quasinormal modes. (QNMs)

QUASI $\Rightarrow$ because the problem is dissipative (non-self-adjoint) $\Rightarrow$ $\underbrace{\omega \in \mathbb{R}}_{\text{NORMAL MODES}} \rightsquigarrow \underbrace{\omega \in \mathbb{C}}_{\text{QUASINORMAL}}$

QNMs are labelled as:

$\omega_{lmn}$

$l, m$ multipolar indices

$n$ overtone ($n=0$ is the least damped, i.e., longest living frequency given $l$ & $m$).

QNMs tells us about the mode stability of the BH.

$$\omega \in \mathbb{C} \rightsquigarrow \omega = \omega_R + i\omega_I \quad \therefore \quad e^{-i\omega t} = e^{-i(\omega_R + i\omega_I)t}$$

$$= \underbrace{e^{-i\omega_R t}}_{\text{OSCILLATION}} \; \underbrace{e^{+\omega_I t}}_{\text{GROWTH } (\omega_I > 0),\ \text{DECAY } (\omega_I < 0)}$$

Some numbers? In units where $r_H/c = \frac{2MG}{c^2} \cdot \frac{1}{c} = 1$, we have for a SCHWARZSCHILD black hole an $l=2$ (any $m$) and $n=0$:

$$\omega \simeq 0.747 - i\, 0.178 \qquad (11)$$

Restoring units:

$$f = \frac{\omega_R}{2\pi} \simeq \frac{0.747}{2\pi} \frac{c}{r_H} \simeq 12 \text{ kHz} \left( \frac{M_\odot}{M} \right) \qquad (12)$$

$$\tau := \frac{1}{|\omega_I|} \simeq \frac{1}{0.178} \frac{r_H}{c} \simeq 5.5 \times 10^{-5} \text{ s} \left( \frac{M_\odot}{M} \right)^{-1} \qquad (13)$$

(DAMPING TIMESCALE)

- for $M = 10\, M_\odot$ : $f \simeq 1$ kHz $\tau \simeq$ tens of ms
- for $M = 10^6 M_\odot$ : $f = 10$ mHz $\tau \sim 1$ minute

Exercise: use fitting forumlas (E5) and (E6) in gr-qc/0512160 to explore dependency of (12) and (13) with spin.

Let's analyse $V$ further. Define:

$$Q(x) = \omega^2 - V(x) \qquad (14)$$

and check for the extrema. For brevity, we write:

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}(1-s^2)\right] := \left(1 - \frac{2M}{r}\right)\left(\frac{\Lambda}{r^2} + \frac{2M\sigma}{r^3}\right) \qquad (15)$$

$\sigma := 1 - s^2$
$\Lambda := l(l+1)$

Then:

$$\frac{dQ}{dx} = \underbrace{\frac{dr}{dx}}_{\text{use Eq. (7)}} \frac{dQ}{dr}$$

$$= -\left(1 - \frac{2M}{r}\right)\frac{dV}{dr}$$

$$= \left(1 - \frac{2M}{r}\right)\left[\frac{2\Lambda}{r^3}\left(1 - \frac{3M}{r}\right) + \frac{2M\sigma}{r^4}\left(3 - \frac{8M}{r}\right)\right]$$

Set to zero. $r = 2M$ is an extrema. Another is:

$$r_0 = \frac{3}{2}\frac{M}{\Lambda}\left[\Lambda - \sigma + \left(\sigma^2 + \Lambda^2 + \frac{14}{9}\sigma\Lambda\right)^{1/2}\right] \qquad (16)$$

As $\ell \to \infty$, i.e., $\Lambda \to \infty$:

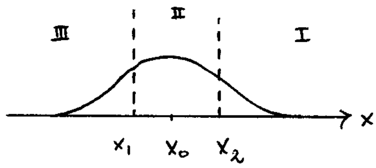
$$r_0 \sim \frac{3}{2}\frac{M}{\Lambda}(2\Lambda) = 3M \qquad (\text{no } \sigma\text{, i.e., } s \text{ dependence})$$

What is special for $r = 3M$ in Schwarzschild? Light ring $r_{LR}$. $x(r_{LR}) =$ M

Is there a relation between QNMs and null geodesics as $\ell \to \infty$?

Goebel 1972

As $\ell \to \infty$, $V$ becomes more peaked (amplitude increases) and peaks closer and closer to $3M$. Use Wentzel-Brillouin-Kramers (WKB) analysis [Schutz & Will 1985]. Consider:




$$\frac{d^2\psi}{dx^2} + Q(x)\,\psi = 0 \qquad (17)$$

Eq. (8) + (14)

$$\psi_{III} \sim Q^{-1/4} \exp\left\{\pm i \int_x^{x_1} Q^{1/2}(x')\,dx'\right\} \qquad \psi_I \sim Q^{1/4} \exp\left\{\pm i \int_{x_2}^{x} Q^{1/2}(x')\,dx'\right\}$$

Match $\psi_{III}$ and $\psi_I$ with a solution for region II, to obtain a Bohr-Sommerfeld quantization condition. The turning points are at $Q(x) = 0$ i.e., $\omega^2 = V(x)$. WKB works well when $\omega^2 \sim V_{peak}$. In region II, we then expand

$$Q(x) \simeq Q_0 + \underbrace{\frac{1}{2} Q_0''}_{:=\,k} (x - x_0)^2 + \dots \qquad Q_0 = Q(r_0), \quad Q_0'' = \left.\frac{d^2 Q}{dx^2}\right|_{x = x_0}$$

Do change of variables

$$t = (4k)^{1/4} e^{i\pi/4} (x - x_0)$$

Then (17) becomes:

$$\frac{d^2\psi}{dt^2} + \Big[\underbrace{-i \frac{Q_0}{(2Q_0'')^{1/2}}}_{:=\,\nu + \frac{1}{2}} - \frac{1}{4} t^2\Big]\psi = 0$$

$$\frac{d^2\psi}{dt^2} + \left[\nu + \frac{1}{2} - \frac{1}{4} t^2\right]\psi = 0 \qquad (\text{parabolic cylindrical function})$$

Analyzing the equation and imposing the QNM boundary condition forces $\nu$ to be an integer, so:

$$i\left(n+\frac{1}{2}\right) = \frac{Q_0}{(2Q_0'')^{1/2}} \qquad n = 0, 1, \dots \qquad (18)$$

We know that

$Q_0 = \omega^2 - V(r_0)$ [with $r_0$ given by Eq. (16)]

$Q_0''$ [can be computed analytically]

NB.: "latest" higher-order WKB formulae: R. KONOPLYA et al. 1904.10333

As $l \to \infty$, (18) reduces to:

$$\omega M = \frac{1}{3\sqrt{3}}\, l - i\left(n+\frac{1}{2}\right)\frac{1}{3\sqrt{3}}$$

$$\omega \simeq \underbrace{\Omega_c}_{\substack{\text{angular frequency} \\ \text{at } r=3M}}\, l - i\left(n+\frac{1}{2}\right)\underbrace{|\lambda_0|}_{\substack{\text{Lyapunov exponent} \\ \text{of geodesics at} \\ r \simeq 3M}}$$

GOEBEL (1972): "A distribution of particles in orbits close to the circular orbit expands away from $r = 3M$ with an e-folding time equal to $3 \times 3^{1/2} M$, and so its population near $r = 3M$ decays with time like $\exp\left[-t/(3\times 3^{1/2} M)\right]^n$.

# LECTURE II

In LECTURE I, we postulated the REGGE-WHEELER equation (1957)

$$\frac{d^2\psi}{dx^2} + [\omega^2 - V(r)]\,\psi = 0 \tag{19}$$

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]$$

In this LECTURE II, we will derive it.

approaches to BHPT $\begin{cases} \text{'metric perturbations'} & \rightarrow \text{ will do} \\ \text{'curvature perturbations'} & \rightarrow \text{ will give an overview of.} \end{cases}$

Continue studying a static spherically symmetric spacetime. Split coordinates as:

$$x^\mu = (t, r, \vartheta, \varphi) \rightsquigarrow x^\mu = (\overset{y^a}{\cancel{z^A}}, \overset{z^A}{\cancel{y^a}}) \tag{20}$$

$$z^A = \{\vartheta, \varphi\} \qquad y^a = \{t, r\}$$

$A, B, C, \dots$ for ~~nonangular~~ angular components

$a, b, c, \dots$ for non angular components

Introduce the metric on the unit two-sphere:

$$ds^2 = \gamma_{AB}\, dz^A\, dz^B = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \tag{21}$$

i.e.,

$$\gamma_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix} \tag{22}$$

and define the LEVI-CIVITA symbol and pseudotensor

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \varepsilon_{AB} = \underbrace{\sqrt{\gamma}}_{\det\gamma}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sin\vartheta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

What is this good for? It allows us, e.g., to write the scalar spherical harmonic equation as:

$$\gamma^{AB}\,\nabla_A\nabla_B\, Y_{\ell m}(z^C) = -\ell(\ell+1)\, Y_{\ell m}(z^C) \tag{23}$$

The idea is to extend this construction for <u>tensor</u> and <u>vector</u> harmonics.

Start from

$$g_{\mu\nu} = \underbrace{g^{(0)}_{\mu\nu}}_{\substack{\text{could be} \\ \text{SCHWARZSCHILD}}} + \underbrace{h_{\mu\nu}}_{\text{perturbation}} \qquad (24)$$

$h_{\mu\nu} = h_{\nu\mu}$ $\therefore$ 10 components in general. $h_{\mu\nu}$ has the structure:

$$h_{\mu\nu} = \begin{pmatrix} \boxed{h_{ab}} & \boxed{h_{aB}} \\ \boxed{h_{Ab}} & h_{AB} \end{pmatrix}$$

scalar; vector; vector; tensor

$$h_{ab} = \begin{pmatrix} h_{00} & h_{01} \\ \underbrace{h_{10}}_{= h_{01}} & h_{11} \end{pmatrix} \Rightarrow 3 \text{ scalars} \qquad (25)$$

$$h_{aB} = ( h_{02} , h_{03} )$$

$h_{Ab} = ( h$

We can decompose each block according to how they transform under a parity transformation

$$(\vartheta, \varphi) \to (\pi - \vartheta, \pi + \varphi) \qquad (26)$$

For example:

$$Y_{\ell m} (\pi - \vartheta, \pi + \varphi) = (-1)^{\ell} \, Y_{\ell m} (\vartheta, \varphi) \qquad (27)$$

If we pick:

$(-1)^{\ell}$ "even parity" (or "polar", or "electric")

$(-1)^{\ell+1}$ "odd parity" (or "axial", or "magnetic")

Just as $Y_{\ell m}$ forms an orthonormal basis to expand scalar function on the sphere, we construct vector and tensor harmonics to expand vector and tensor fields on the sphere.

VECTOR

POLAR : $$Y_A^{\ell m} := \nabla_A Y^{\ell m} = ( \partial_\vartheta Y^{\ell m} , \partial_\varphi Y^{\ell m} ) \qquad (28)$$

AXIAL : $$S_A^{\ell m} := \varepsilon_{AC} \, \gamma^{BC} \, \nabla_B Y^{\ell m} = \left( \frac{1}{\sin\vartheta} \partial_\varphi Y^{\ell m} , - \sin\vartheta \, \partial_\vartheta Y^{\ell m} \right)$$

TENSOR

POLAR : $$Z_{AB}^{\ell m} := \nabla_A \nabla_B Y^{\ell m} + \frac{1}{2} \ell(\ell+1) \gamma_{AB} Y^{\ell m} = \frac{1}{2} \begin{pmatrix} W^{\ell m} & X^{\ell m} \\ X^{\ell m} & -\sin^2\vartheta \, W^{\ell m} \end{pmatrix}$$

AXIAL : $$S_{AB}^{\ell m} := \frac{1}{2} \left( \nabla_B S_A^{\ell m} + \nabla_A S_B^{\ell m} \right) = \frac{1}{2} \begin{pmatrix} \sin^{-1}\vartheta \, X^{\ell m} & -\sin\vartheta \, W^{\ell m} \\ -\sin\vartheta \, W^{\ell m} & -\sin\vartheta \, X^{\ell m} \end{pmatrix} \qquad (29)$$

where

$$X^{\ell m} := 2 \left( \partial_{\vartheta\varphi} Y^{\ell m} - \cot\vartheta \, \partial_\varphi Y^{\ell m} \right)$$

$$W^{\ell m} := \partial_{\vartheta\vartheta} Y^{\ell m} - \cot\vartheta \, \partial_\vartheta Y^{\ell m} - \sin^{-2}\vartheta \, \partial_{\varphi\varphi} Y^{\ell m} \qquad (30)$$

Lots of definitions. But now we can write:

$$h_{ab}(y^a, z^A) \sim \bar{h}_{ab}^{\ lm}(y^a)\ Y^{lm}(z^A)$$

" like $\psi/r\ Y_{lm}$ " in LECTURE I

$$h_{aA}(y^a, z^A) \sim h^{pol.}_{a,\ lm}(y^a)\ Y_A^{\ lm}(z^A) + h^{ax}_{a,\ lm}(y^a)\ S_A^{\ lm}(z^A)$$

$$h_{AB}(y^a, z^A) \sim r^2 \left\{ K_{lm}(y^a)\ \gamma_{AB}\ Y^{lm}(z^A) + G_{lm}(y^a)\ \nabla_A \nabla_B Y^{lm}(z^A) + 2\ h_{lm}(y^a)\ S_{AB}(z^A) \right\} \tag{31}$$

" SO(2) irreducible representation "

We can define inner products

$$\langle f, g \rangle = \int \underset{\uparrow\ \sin^2\vartheta\, d\vartheta\, d\varphi}{d\Omega}\ f^* g$$

$$\langle f_A, g_A \rangle = \int d\Omega\ \gamma^{AB}\ f_A^*\ g_B \tag{32}$$

$$\langle f_{AB}, g_{AB} \rangle = \int d\Omega\ \gamma^{AC}\ \gamma^{BD}\ f^*_{AB}\ g_{CD}$$

using that $\langle Y^{lm}, Y^{l'm'} \rangle = \delta^{ll'}\ \delta^{mm'}$

we have:
$$\langle Y_A^{\ lm}, Y_A^{\ l'm'} \rangle = \langle S_A^{\ lm}, S_A^{\ l'm'} \rangle = \underset{\delta^{ll'}\delta^{mm'}}{l(l+1)} := 2(n+1)\ \delta^{ll'}\ \delta^{mm'}$$

$$n = (l+2)(l-1)/2 \tag{33}$$

and:
$$\langle Z_{AB}^{\ lm}, Z_{AB}^{\ l'm'} \rangle = \langle S^{\ lm}_{AB}, S^{\ l'm'}_{AB} \rangle = 2n(n+1)\ \delta^{ll'}\ \delta^{mm'}$$

The are the normalizations.

We can eliminate some functions by a gauge choice. In the REGGE-WHEELER gauge

$$h_{a,\,lm}^{pol} = G_{lm} = h_{lm} = 0 \qquad (34)$$

such that

$$h_{\mu\nu} = \begin{pmatrix} \bar{h}_{ab,\,lm} \, Y^{lm} & h_{a,\,lm}^{ax} \, S_A^{\,lm} \\ h_{a,\,lm}^{ax} & r^2 K_{lm} \, \gamma_{AB} \, Y^{lm} \end{pmatrix} \qquad (35)$$

$h_{a,\,lm}^{ax} =$ "$h_a$" from now on

with:

$$\bar{h}_{ab,\,lm} = \begin{pmatrix} H_{0,\,lm} & H_{1,\,lm} \\ H_{1,\,lm} & H_{2,\,lm} \end{pmatrix} \qquad \text{3 scalar functions}$$

$$h_{a,\,lm} = \begin{pmatrix} h_{0,\,lm} \\ h_{1,\,lm} \end{pmatrix} \qquad \text{vector (axial)}$$

$$K \qquad \text{scalar}$$

$$\underset{\substack{\uparrow \\ \text{RW gauge}}}{10 - 4} = 6 \begin{cases} \nearrow \ 2 \text{ axial vars. } (h_0, h_1) \\ \searrow \ 4 \text{ polar vars. } (H_0, H_1, H_2, K) \end{cases}$$

PHEW! From now on we omit "$lm$" and will vacuum EINSTEIN eqs. ($R_{\mu\nu} = 0$)

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

$$R_{\mu\nu}[g_{\mu\nu}] \sim \underset{\substack{\| \\ 0}}{R_{\mu\nu}[g^{(0)}_{\mu\nu}]} + \underbrace{\delta R_{\mu\nu}[h_{\mu\nu}]}_{\text{this is what we want}} + \mathcal{O}(h^2) \qquad (36)$$

Under perturbation CHRISTOFFEL becomes:

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left( h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu} \right) \qquad (37)$$

and the Ricci tensor becomes:

$$\delta R_{\mu\nu} = - \delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu} \qquad (38)$$

$;$ := covariant derivatives with respect to $g^{(0)}$

Because of spherical symmetry <u>axial</u> and <u>polar</u> sectors can be analysed separately.

We will focus on the axial case. We take all variables and write

$$f(y^a) = f(r)\, e^{-i\omega t}$$

For the axial perturbations, only the $t\varphi$, $r\varphi$, $\vartheta\varphi$ components are nontrivial. $t\varphi$ can be obtained from $r\varphi$ and $\vartheta\varphi$ (i.e., it is redundant).

$$r\varphi : \quad 2ir^2\omega h_0 + [(r-2M)(2-\ell-\ell^2) + r^3\omega^2] h_1 - ir^3\omega h_0' = 0 \quad (38)$$

$$\vartheta\varphi : \quad ir^3\omega h_0 + (r-2M)[2M h_1 + (r-2M) r h_1'] = 0 \quad (39)$$

$' := d/dr$

Now, solve (39) for $h_0$:

$$h_0 = \frac{i}{\omega r^2}\left(1 - \frac{2M}{r}\right)[2M h_1 + r(r-2M) h_1'] \quad (40)$$

substitute in (38) and define:

$$\psi^{RW} := \frac{1}{r}\left(1 - \frac{2M}{r}\right) h_1 \quad (41)$$

Introducing tortoise coordinates we find:

$$\boxed{\frac{d^2\psi^{RW}}{dx^2} + \left\{\omega^2 - \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]\right\}\psi^{RW} = 0} \quad (42)$$

NOTES:

(i) once we know $\psi^{RW}$, $h_1$ can be found from (41) and $h_0$ from (40)

(ii) the polar case is much more involved. ZERILLI (1970)

$$\frac{d^2\psi^Z}{dx^2} + \left\{\omega^2 - \left(1 - \frac{2M}{r}\right)\frac{1}{r^3}\left[\frac{2n^2(n+1)r^3 + 6n^2Mr + 18nM^2r + 18M^3}{(nr+3M)^2}\right]\right\}\psi^Z = 0$$

$n := (\ell+2)(\ell-1)/2$

cf.: SAGO-NAKANO-SASAKI PRD 67, 104017 (2013)

For $d>4$ BHs; KODAMA-ISHIBASHI (2003) hep-th/0305147

The REGGE-WHEELER and ZERILLI potentials are analytically very different, yet they are quite similar.

$$\underbrace{\psi^{RW}\,,\quad \psi^{Z}}_{2\ \text{DOF}}$$

The QNMs of each equation are the same. (isospectral)

CHANDRASEKHAR '80, " - DETWEILER '85

See also: GLAMPEDAKIS, JOHNSON, KENNEFICK 1702.06459

(iii) Sources. In general, we could have a $T_{\mu\nu}$, say of a test particle driving the perturbations.

$$\frac{d^2\psi^{RW/Z}}{dx^2} + \left\{ \omega^2 - V^{RW/Z} \right\} \psi^{RW/Z} = S^{RW/Z}$$

↑ follow same reductions, ~~but~~ now decomposing

$$T^{\mu\nu}(x^\alpha) = \mu \int \frac{d\tau}{\sqrt{-g}}\, u^\mu(\tau)\, u^\nu(\tau)\, \delta^4[x - x_p(\tau)]$$

into harmonics.

(iv) GWs: here we worked in the RW gauge. How do we relate $\psi^{RW/Z}$ with the more familiar $h_{+,\times}$? Qualitatively:

(1) Introduce a tetrad for stationary observers in SCHWARZSCHILD.

(2) Project $h$ into tetrad and evaluate it far from BH.

(3) Impose traceless-ness of $h$.

$$h_+ - i h_\times = \frac{1}{r} \sum_{\ell m} \sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\, \left( \psi^{Z} + i \psi^{RW} \right) {}_{-2}Y^{\ell m}(\vartheta,\varphi) + \mathcal{O}(r^{-2})$$

NB.: ${}_{-2}Y^{\ell m}(\vartheta,\varphi)$ are the spin weighted spherical harmonics

$${}_{-2}Y^{\ell m}(\vartheta,\varphi) = \sqrt{\frac{(\ell-2)!}{(\ell+2)!}} \left( W^{\ell m}(\vartheta,\varphi) - \frac{i}{\sin\vartheta} X^{\ell m}(\vartheta,\varphi) \right)$$

(v) Teukolsky equation.

<u>Idea</u>: introduce null tetrads (Newman-Penrose)

Given $g_{\mu\nu}$ the null tetrads are 4 linearly independent four-vectors

$$z_a^{\mu} := ( \underbrace{l^{\mu}, n^{\mu}}_{\text{real}}, \underbrace{m^{\mu}, \bar{m}^{\mu}}_{\text{complex}} )$$

complex; "$-$" complex conjugation

By <u>null</u>, we mean that:

$$g_{\mu\nu} l^{\mu} l^{\nu} = g_{\mu\nu} n^{\mu} n^{\nu} = g_{\mu\nu} m^{\mu} m^{\nu} = g_{\mu\nu} \bar{m}^{\mu} \bar{m}^{\nu} = 0$$

Furthermore:

$$g_{\mu\nu} m^{\mu} \bar{m}^{\nu} = 1 \qquad g_{\mu\nu} l^{\mu} n^{\nu} = -1 \qquad g_{\mu\nu} l^{\nu} = l_{\mu}$$

for instance

all other products are zero. Then:

$$g^{\mu\nu} = m^{\mu} \bar{m}^{\nu} + m^{\nu} \bar{m}^{\mu} - l^{\mu} n^{\nu} - l^{\nu} n^{\mu}$$

The four-vectors form a basis in which we can expand any tensor on:

$$R_{abcd} = z_a^{\mu} z_b^{\nu} z_c^{\rho} z_d^{\sigma} R_{\mu\nu\rho\sigma}$$

For example, Minkowski looks like:

$$l^{\mu} = (1, 0, 0, 1)$$
$$n^{\mu} = \tfrac{1}{2} (1, 0, 0, -1)$$
$$m^{\mu} = \frac{1}{\sqrt{2}} (0, 1, i, 0)$$
$$\bar{m}^{\mu} = \frac{1}{\sqrt{2}} (0, 1, -i, 0)$$

$\mu = (t, x, y, z)$

Then, e.g.:

$$\eta^{00} = \cancel{m^0 \bar{m}^0}^{\,0} + \cancel{m^0 \bar{m}^0}^{\,0} - l^0 n^0 - l^0 n^0$$
$$= -\frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1 = -1$$

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

For a Kerr BH, a popular null tetrad, is that of Kinnersley.

In Boyer-Lindquist coordinates:

$$l^{\mu} = \frac{1}{\Delta} ( r^2 + a^2, \Delta, 0, a ) \qquad \rho^2 := r^2 + a^2 \cos^2\vartheta$$

$$n^{\mu} = \frac{1}{2\rho^2} ( r^2 + a^2, -\Delta, 0, a ) \qquad \Delta := r^2 - 2Mr + a^2$$

$$m^{\mu} = \frac{1}{\sqrt{2}} \frac{1}{r + i a \cos\vartheta} \left( i a \sin\vartheta, 0, 1, \frac{i}{\sin\vartheta} \right)$$

$\mu = (t, r, \vartheta, \varphi)$

The key quantities in the NEWMAN-PENROSE formalism are certain projections of the WEYL tensor onto a null tetrad.

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left( g_{\mu\rho} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} + g_{\nu\sigma} R_{\mu\rho} \right) + \frac{1}{6} R \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

Properties:

(1) vanishes for any contraction of two indices (the trace-free part of RIEMANN).

(2) same symmetries of the RIEMANN tensor

$$C_{\mu\nu\rho\sigma} = - C_{\nu\mu\rho\sigma}$$
$$C_{\mu\nu\rho\sigma} = - C_{\mu\nu\sigma\rho}$$
$$C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu}$$
$$C_{\mu\nu\rho\sigma} + C_{\mu\sigma\nu\rho} + C_{\mu\rho\sigma\nu} = 0$$

(3) In $\underline{4D}$, 10 independent components.

The NEWMAN-PENROSE scalars are:

"WEYL scalars"

$$\Psi_0 = C_{\mu\nu\rho\sigma}\, l^\mu m^\nu l^\rho m^\sigma$$
$$\Psi_1 = C_{\mu\nu\rho\sigma}\, l^\mu k^\nu l^\rho m^\sigma$$
$$\Psi_2 = (1/2)\, C_{\mu\nu\rho\sigma}\, l^\mu l^\nu \left( l^\rho k^\sigma + m^\rho \bar{m}^\sigma \right)$$
$$\Psi_3 = C_{\mu\nu\rho\sigma}\, k^\mu l^\nu k^\rho \bar{m}^\sigma$$
$$\Psi_4 = C_{\mu\nu\rho\sigma}\, k^\mu \bar{m}^\nu k^\rho \bar{m}^\sigma$$

$\Psi_{0,\dots,4} \in \mathbb{C}$

$\therefore\ 2 \times 5 = \underline{10}$

You saw:

$$R_{0i0j} = -\frac{1}{2c^2}\, \ddot{h}^{TT}_{ij}$$

GW propagating in flat space along $z$-direction.

$$\Downarrow$$

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}$$

In vacuum $R_{\mu\nu} = 0$, so $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$.. We can now use our Cartesian null tetrad & and $R_{0i0j}$ to compute the WEYL scalars.

We find

$$\psi_4 = -\frac{1}{8c^2}\left(\ddot{h}_+ - i\ddot{h}_\times\right) \quad , \quad \psi_0 = -\frac{1}{2c^2}\left(\ddot{h}_+ + \ddot{h}_\times\right)$$

$$\psi_1 = \psi_2 = \psi_3 = 0$$

$$\psi_0 \sim \psi_4^*$$

So all radiative info about GWs are in $\psi_4$ $(\in \mathbb{C})$.

The idea of the TEUKOLSKY formalism is to:

(i) fix a background geometry by choosing $l^\nu_A, n^\nu_A, \bar{m}^\nu_A, m^\nu_A$

(ii) perturb them.

(iv) propagate "linearization" into all geometric objects.

(v) find an eq. for perturbed $\psi_{0,4}$.

$$\mathcal{L}_T\left[{}_s\psi_{\ell m}\right] = 0$$

$\uparrow$ spin weight

$s = +2 \qquad \psi = \psi_0$

$s = -2 \qquad \psi = (r - ia\cos\theta)\psi_4$

TEUKOLSKY eq WITH $a = 0$ is known as BARDEEN-PRESS equation.

The QNMs of the BARDEEN-PRESS equation are isospectral to the REGGE-WHEELER and ZERILLI equations; expected, but not obvious!

In freq. domain the BARDEEN-PRESS-TEUKOLSKY eqs. with sources can be cumbersome to solve ( V is long-ranged and complex valued). An alternative eq. was proposed by SASAKI-NAKAMURA (1982).

NB.: it is a "miracle" that the TEUKOLSKY equation exists at all. See the opening paragraph of "sec. IIIB of" GLAMPEDAKIS CQG 22, 5605 (2005)