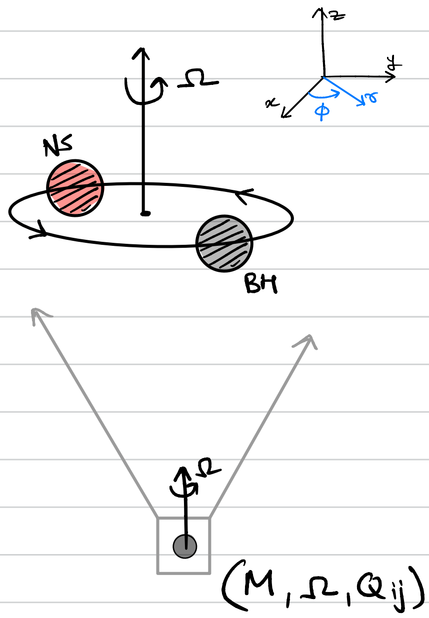
Gravitational-wave Course

Instructor: Alessandra Buonanno

Tutor: Raj Patil
Tutor: Lorenzo Pompili

Solutions to HW6

I. Newtonian quadrupolar tidal imprint in GW phasing



⊛ System: Neutron star (NS) and Black hole (BH) binary of total mass M

Where the dynamics is given by Lagrangian,

$$L = \frac{1}{2} m_1 \dot{\bar{r}}_1^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2^2 - V(|\bar{r}_1 - \bar{r}_2|)$$

where, $\bar{r}_1$ is the position of BH
$\bar{r}_2$ is the position of NS

lets first convert the problem from two body to one body by going to center-of-mass co-ordinates, defined by $m_1 \bar{r}_1 + m_2 \bar{r}_2 = \bar{0}$.
Here we also shifted the center of mass to the origin $\bar{0}$.

$$\Rightarrow \quad \bar{r}_1 = \frac{m_2 \bar{r}}{m_1 + m_2} \quad \text{and} \quad \bar{r}_2 = \frac{-m_1 \bar{r}}{m_1 + m_2} \quad \text{where} \quad \bar{r} = \bar{r}_1 - \bar{r}_2 .$$

Thus the Lagrangian becomes,

$$L = \frac{1}{2} m_1 \left(\frac{m_2 \dot{\bar{r}}}{m_1 + m_2}\right)^2 + \frac{1}{2} m_2 \left(\frac{-m_1 \dot{\bar{r}}}{m_1 + m_2}\right)^2 - V(|\bar{r}|)$$

$$= \frac{1}{2} \frac{m_1 m_2^2}{(m_1 + m_2)^2} \dot{\bar{r}}^2 + \frac{1}{2} \frac{m_2 m_1^2}{(m_1 + m_2)^2} \dot{\bar{r}}^2 - V(|\bar{r}|)$$

$$= \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2}\right) \dot{\bar{r}}^2 \left[\frac{m_2}{m_1 + m_2} + \frac{m_1}{m_1 + m_2}\right] - V(|\bar{r}|)$$

$$= \frac{1}{2} \mu \dot{\bar{r}}^2 - V(|\bar{r}|) \qquad \text{where} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \equiv \text{reduced mass}$$

Converting the above in polar-co-ordinates (r, ϕ, r_z)

where,

$$\bar{r} \equiv (r_x, r_y, r_z) \quad ; \quad r = \sqrt{r_x^2 + r_y^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{r_y}{r_x}\right)$$

$$r_x = r \cos\phi$$

$$r_y = r \sin\phi$$

$$\dot{r}_x = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$$
$$\dot{r}_y = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$

$$\bar{r}^2 = r^2 + r_z^2$$
$$\dot{\bar{r}}^2 = \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 = \dot{r}^2 + r^2\dot{\phi}^2 + r_z^2$$

Now, assuming that the binary is moving in x-y plane and thus $r_z = 0$. In polar-co-ordinates the Lagrangian is given by

$$\boxed{L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)}$$

where for the potential $V(r)$ we known,

1. the gravitational effects of mass monopole are described by Newtonian potential

$$V_M(r) = -\frac{m_1 m_2}{r} = \frac{-\mu M}{r}$$ where $M = m_1 + m_2$ and we set $G = 1$

2. the effects of Newtonian tidal field on response quadrupole moment are given by

$$V_Q(r) = +\frac{1}{2} Q^{ij}\mathcal{E}_{ij}$$

where $\mathcal{E}_{ij} \equiv$ Newtonian tidal field

$$= -m_{BH}\,\partial_i\partial_j\left(\frac{1}{r}\right)$$

$$= -m_{BH}\frac{\left(3n^i n^j - \delta^{ij}\right)}{r^3}$$

where $n^i = \frac{r^i}{r}$ and $i = x, y, z$

$n^i n_i = 1$ and $\delta^{ij}\delta_{ij} = 3$

3. the potential energy of conservative response quadrupole moment

$$V_{int}(r) = +\frac{1}{4\lambda} Q_{ij} Q^{ij}$$

Here we compute dimensions of coupling constants appearing above.

Now $c = 1 \Rightarrow L = T$, $G = 1 \Rightarrow L^3 = T^2 M$ $\Big\}\Rightarrow M = T = L$ So we measure mass dimensions.

In this, using equation ①, $[\lambda] = M^5$

Now we make a crucial assumption,

<u>Assumption</u>: Quadrupole is adiabatically induced $Q_{ij} = -\lambda \mathcal{E}_{ij}$
where λ is deformability parameter.

$$\Rightarrow \quad V_Q(r) = +\frac{1}{2}\left(-\lambda \mathcal{E}^{ij}\right)\mathcal{E}_{ij}$$

$$= -\frac{1}{2}\lambda \mathcal{E}^{ij}\mathcal{E}_{ij} = -\frac{1}{2}\lambda M_{BH}^2 \frac{\left(3n^i n^j - \delta^{ij}\right)}{r^3}\frac{\left(3n_i n_j - \delta_{ij}\right)}{r^3}$$

$$= -\frac{1}{2}\lambda \frac{M_{BH}^2}{r^6}(9-3-3+3) = -3\lambda \frac{M_{BH}^2}{r^6}$$

$$V_{int}(r) = \frac{1}{4\lambda}\left(-\lambda \mathcal{E}_{ij}\right)\left(-\lambda \mathcal{E}^{ij}\right) = \frac{\lambda}{4}\left(\mathcal{E}^{ij}\mathcal{E}_{ij}\right)$$

$$= \frac{\lambda}{4}\frac{M_{BH}^2\, 6}{r^6} = \frac{3}{2}\lambda \frac{M_{BH}^2}{r^6}$$

Hence,

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V_M(r) - V_Q(r) - V_{int}(r)$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{\mu M}{r} + 3\lambda \frac{M_{BH}^2}{r^6} - \frac{3}{2}\lambda \frac{M_{BH}^2}{r^6}$$

$$\boxed{L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{\mu M}{r} + \frac{3}{2}\lambda \frac{M_{BH}^2}{r^6}} \qquad (1)$$

ⓐ **Equation of motion**:

Using Euler-Lagrange equations on Lagrangian given in ①, equation of motion for r is given by $\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$

$$\frac{\partial L}{\partial r} = \frac{1}{2}\mu\, 2r\dot{\phi}^2 + \left(\frac{-1}{r^2}\right)\mu M + \left(\frac{-6}{r^7}\right)\frac{3}{2}\lambda m_{BH}^2$$

$$= \mu r\dot{\phi}^2 - \frac{\mu M}{r^2} - \frac{9\lambda m_{BH}^2}{r^7}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \mu\ddot{r}$$

$$\Rightarrow \quad \boxed{-\mu\ddot{r} + \mu r\dot{\phi}^2 - \frac{\mu M}{r^2} - \frac{9\lambda m_{BH}^2}{r^7} = 0} \quad \text{———} \quad ②$$

and equation of motion for ϕ is given by $\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}\mu r^2\, 2\dot{\phi} \quad ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \mu\left(r^2\ddot{\phi} + 2r\dot{r}\dot{\phi}\right)$$

And since $r > 0$ in our system,

$$\Rightarrow \quad \boxed{r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0} \quad \text{———} \quad ③$$

(b) **Linear tidal correction to radius:**

Assumption: orbit is circular $\Rightarrow \ddot{r} = \dot{r} = 0$ and $\dot{\phi} = \Omega$.

$$(2) \Rightarrow \mu r \Omega^2 - \frac{\mu M}{r^2} - \frac{9\lambda m_{BH}^2}{r^7} = 0$$

$$(3) \Rightarrow \dot{\Omega} = 0 \Rightarrow \Omega \text{ is constant. let } \Omega(t) = \Omega$$

hence,

$$\mu r \Omega^2 - \frac{\mu M}{r^2} - \frac{9\lambda m_{BH}^2}{r^7} = 0$$

$$\Rightarrow \Omega^2 - \frac{M}{r^3} - \frac{9\lambda}{r^8} \cdot \frac{m_{BH}^2}{\mu} = 0$$

We want the answer in the following form $r(\Omega) = \left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta r)$

So let's substitute this form in the equation above and solve for δr

$$\Rightarrow \Omega^2 - \frac{M}{\left[\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta r)\right]^3} - \frac{9\lambda}{\left[\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta r)\right]^8} \frac{m_{BH}^2}{\mu} = 0$$

$$\Rightarrow \Omega_0^2 - \frac{\Omega_0^2}{(1+\delta r)^3} - \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3} \frac{1}{(1+\delta r)^8} = 0$$

Assumption: $\delta r \ll 1$

$$\Rightarrow \Omega^2 - \Omega^2\left[1 - 3\delta r + \mathcal{O}((\delta r)^2)\right] - \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3}\left[1 - 8\delta r + \mathcal{O}((\delta r)^2)\right] = 0$$

Ignoring terms of $\mathcal{O}((\delta r)^2)$,

$$\Rightarrow +3\delta r\,\Omega^2 - \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3} + \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3} 8\,\delta r = 0$$

$$\left(\frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3} 8 + 3\Omega^2\right)\delta r = \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3}$$

$$\delta r = \frac{1}{\left(\frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3} 8 + 3\Omega^2\right)} \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3}$$

$$\delta r = \frac{1}{3\Omega^2\left(1 + 24\left(\frac{M}{\mu}\right)\left(\frac{m_{BH}}{M}\right)^2\left(\frac{\lambda}{M^5}\right)(\Omega M)^{10/3}\right)} \qquad \frac{9\lambda m_{BH}^2}{\mu}\left(\frac{\Omega^2}{M}\right)^{8/3}$$

Assumption: $\frac{M}{\mu}$, $\frac{m_{BH}}{M}$, $(M\Omega)$ are order one and $\frac{\lambda}{M^5} \ll 1$

$$\Rightarrow \boxed{\delta r = \frac{3\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}} \qquad (4)$$

This is a constant change in the radius of circular orbits due to the tidal deformations.

Hence now the orbits are still circular upto order λ, but with radius

$$r(\Omega) = \left(\frac{M}{\Omega^2}\right)^{1/3}(1 + \delta r)$$

(c) Energy of the system:

Lets begin with the Lagrangian,

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V_M(r) - V_Q(r) - V_{int}(r)$$

$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\mu M}{r} - \frac{1}{2} Q^{ij}\mathcal{E}_{ij} - \frac{1}{4\lambda} Q_{ij}Q^{ij}$$

Now computing conjugate momentas for r and ϕ,

$$P_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r} - \frac{1}{2}\mathcal{E}_{ij}\frac{\partial}{\partial \dot{r}}(Q^{ij}) - \frac{1}{2\lambda} Q_{ij}\frac{\partial}{\partial \dot{r}}(Q^{ij})$$

$$= \mu\dot{r} - \frac{1}{2}\frac{\partial}{\partial \dot{r}}(Q^{ij})\left[\mathcal{E}_{ij} + \frac{1}{\lambda}Q_{ij}\right]$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2\dot{\phi} - \frac{1}{2}\mathcal{E}_{ij}\frac{\partial}{\partial \dot{\phi}}(Q^{ij}) - \frac{1}{2\lambda} Q_{ij}\frac{\partial}{\partial \dot{\phi}}(Q^{ij})$$

$$= \mu r^2\dot{\phi} - \frac{1}{2}\frac{\partial}{\partial \dot{\phi}}(Q^{ij})\left[\mathcal{E}_{ij} + \frac{1}{\lambda}Q_{ij}\right]$$

Now computing the Hamiltonian

$$H = \dot{r}P_r + \dot{\phi}P_\phi - L$$

$$= \dot{r}\left\{\mu\dot{r} - \frac{1}{2}\frac{\partial}{\partial \dot{r}}(Q^{ij})\left[\mathcal{E}_{ij} + \frac{1}{\lambda}Q_{ij}\right]\right\}$$

$$+ \dot{\phi}\left\{\mu r^2\dot{\phi} - \frac{1}{2}\frac{\partial}{\partial \dot{\phi}}(Q^{ij})\left[\mathcal{E}_{ij} + \frac{1}{\lambda}Q_{ij}\right]\right\}$$

$$- \left\{\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\mu M}{r} - \frac{1}{2}Q^{ij}\mathcal{E}_{ij} - \frac{1}{4\lambda}Q_{ij}Q^{ij}\right\}$$

Now assuming

adiabatic quadrupoles $Q_{ij} = -\lambda\mathcal{E}_{ij} \Rightarrow \frac{\partial Q_{ij}}{\partial \dot{r}} = 0$ and $\frac{\partial Q_{ij}}{\partial \dot{\phi}} = 0$,

and circular orbits $\Rightarrow \ddot{r} = 0 = \dot{r}$ and $\dot{\phi} = \Omega$,

$$\Rightarrow \quad H = \mu\dot{r}^2 + \mu r^2\dot{\phi}^2 - \left[\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\mu M}{r} + \frac{3}{2}\lambda\frac{m_{BH}^2}{r^6}\right]$$

$$H = \frac{1}{2}\frac{P_\phi^2}{\mu r^2} - \frac{\mu M}{r} - \frac{3}{2}\lambda\frac{m_{BH}^2}{r^6}$$

Now substituting the value of $\gamma(\Omega)$ using equation 4,

$$H = \frac{1}{2}\mu\left[\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta\gamma)\right]^2\Omega^2 - \frac{\mu M}{\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta\gamma)} - \frac{3}{2}\frac{\lambda\, m_{BH}^2}{\left[\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta\gamma)\right]^6}$$

$$= \frac{1}{2}\mu\left(\frac{M}{\Omega^2}\right)^{2/3}(1+2\delta\gamma)\,\Omega^2 - \frac{\mu M}{\left(\frac{M}{\Omega^2}\right)^{1/3}}(1-\delta\gamma)$$
$$- \frac{3}{2}\frac{\lambda m_{BH}^2}{\left(\frac{M}{\Omega^2}\right)^2}(1-6\,\delta\gamma) + \mathcal{O}[(\delta\gamma)^2]$$

$$= \frac{1}{2}\mu\Omega^2\left(\frac{M}{\Omega^2}\right)^{2/3}\left(1+\frac{6\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}\right) - \mu M\left(\frac{\Omega^2}{M}\right)^{1/3}\left(1-\frac{3\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}\right)$$
$$-\frac{3}{2}\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2\left(1-18\frac{\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}\right) + \mathcal{O}(\lambda^2)$$

$$= \frac{1}{2}\mu\Omega^2\left(\frac{M}{\Omega^2}\right)^{2/3}\left(1+\frac{6\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}\right) - \mu M\left(\frac{\Omega^2}{M}\right)^{1/3}\left(1-\frac{3\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}\right)$$
$$-\frac{3}{2}\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 + \mathcal{O}(\lambda^2)$$

$$= \frac{1}{2}\mu\Omega^2\left(\frac{M}{\Omega^2}\right)^{2/3} - \mu M\left(\frac{\Omega^2}{M}\right)^{1/3}$$
$$+ 3\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 + 3\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 - \frac{3}{2}\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 + \mathcal{O}(\lambda^2)$$

$$H = -\frac{1}{2}\mu M^{2/3}\Omega^{2/3} + \frac{9}{2}\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 + \mathcal{O}(\lambda^2)$$

Hence the energy is given by

$$\boxed{E = -\frac{1}{2}\mu M^{2/3}\Omega^{2/3} + \frac{9}{2}\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 + \mathcal{O}(\lambda^2)} \qquad (5)$$

Orbital effects: befor jumping into the computation of energy flux and phasing due to tidal effects, lets focus on the orbital contributions.

For the orbital contribution we assume,

Assumptions: The binary is made of two point particles at $\bar{r}_1$ and $\bar{r}_2$ of mass m_1 and m_2, moving in circular orbit.

In center of mass frame and cartesian co-ordinates

$$r_x(t) = r\cos(\Omega t) \qquad \text{where } \dot{\phi} = \Omega$$
$$r_y(t) = r\sin(\Omega t)$$

Now the Quadrupole moment for this system is given by $I_{ij} = \mu r_i r_j$

whose components are given by, $I_{11} = \frac{1}{2}\mu r^2 [1+\cos(2\Omega t)]$

$$I_{12} = I_{21} = \frac{1}{2}\mu r^2 \sin(2\Omega t)$$

$$I_{22} = \frac{1}{2}\mu r^2 [1-\cos(2\Omega t)]$$

$$I_{i3} = 0$$

writing in a matrix form,

$$[I_{ij}] \equiv \begin{bmatrix} 1+\cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1-\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\frac{1}{2}\mu r^2\right)$$

Computing reduced quadrupole moment,

$$Q_{ij}^{(orb)} = I_{ij} - \frac{1}{3}\delta_{ij} I_{kk}$$

$$\equiv \begin{bmatrix} 1+\cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1-\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\frac{1}{2}\mu r^2\right) - \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\frac{1}{2}\mu r^2\right)$$

$$\equiv \left(\frac{1}{2}\mu r^2\right)\frac{1}{3}\begin{bmatrix} 1+3\cos(2\Omega t) & 3\sin(2\Omega t) & 0 \\ 3\sin(2\Omega t) & 1-3\cos(2\Omega t) & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Now we compute Energy flux due to the orbital quadrupole, which is given by,

$$P = +\frac{1}{5}\left\langle \frac{d^3(Q_{ij})}{dt^3}\frac{d^3(Q^{ij})}{dt^3}\right\rangle$$

For this, we need

$$\left[\frac{d^3 Q_{ij}^{(orb)}}{dt^3}\right] \equiv \left(\frac{1}{2}\mu r^2\right)(2\Omega)^3 \begin{bmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

using above in the formula for power,

$$P_{(orb)} = \frac{1}{5}\frac{1}{T}\int_0^T dt \left(4\mu r^2\Omega^3\right)^2 (2)$$

$$= \frac{32}{5}\mu^2 r^4 \Omega^6$$

here we have used $r = \left(\frac{M}{\Omega^2}\right)^{1/3}$ as we have ignored the tidal effects.

$$\boxed{P_{(orb)} = \frac{32}{5}\mu^2 M^{4/3}\Omega^{10/3}}$$

Now computing total energy of the system using the Hamiltonian given in equation ⑤,

$$E_{(orb)} = \frac{-1}{2}\mu M^{2/3}\Omega^{2/3}$$

$$\Rightarrow \quad \frac{dE}{d\Omega} = \frac{-1}{3}\mu M^{2/3}\Omega^{-1/3}$$

Then the gravitational wave phasing can be computed using the stationary phase approximation (SPA) as,

$$\frac{d^2\Psi_{SPA}}{d\Omega^2} = 2\frac{dE/d\Omega}{\dot{E}} = 2\frac{dE/d\Omega}{-P} = 2\frac{\left(\frac{-1}{3}\mu M^{2/3}\Omega^{-1/3}\right)}{\left(\frac{32}{5}\mu^2 M^{4/3}\Omega^{10/3}\right)}$$

$$\boxed{\frac{d^2\Psi_{SPA}}{d\Omega^2} = \frac{10}{96}\frac{M^2}{\left(\frac{\mu}{M}\right)}(M\Omega)^{-\frac{11}{3}}}$$

(d) **Tidal contribution to the energy flux:**

For this we begin by computing quadrupole moment due to tidal effects (using the adiabatic approximation) given by

$$Q_{ij} = -\lambda \mathcal{E}_{ij} = \lambda\, m_{BH} \left(\frac{3 n_i n_j - \delta_{ij}}{r^3} \right)$$

where, $\quad r^i \equiv (r\cos(\Omega t),\ r\sin(\Omega t),\ 0)$

$\qquad\quad n^i \equiv (\cos(\Omega t),\ \sin(\Omega t),\ 0)$

writing the above in matrix form,

$$[Q_{ij}] \equiv \frac{\lambda\, m_{BH}}{r^3} \left[3 \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{pmatrix} \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\equiv \frac{\lambda\, m_{BH}}{r^3} \begin{bmatrix} 3(\cos(\Omega t))^2 - 1 & 3\cos(\Omega t)\sin(\Omega t) & 0 \\ 3\sin(\Omega t)\cos(\Omega t) & 3(\sin(\Omega t))^2 - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\left[\frac{d^3}{dt^3}(Q_{ij}) \right] \equiv \frac{\lambda\, m_{BH}}{r^3} (12\,\Omega^3) \begin{bmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now computing energy flux due to the total quadrupole moment

$$Q_{ij}^{(T)} = Q_{ij\,(orb)} + Q_{ij},$$

$$P = \frac{1}{5} \left\langle \frac{d^3}{dt^3}\left(Q_{ij}^{(T)}\right) \frac{d^3}{dt^3}\left(Q^{ij(T)}\right) \right\rangle$$

$$= \frac{1}{5} \left\langle \frac{d^3}{dt^3}\left(Q_{ij\,(orb)} + Q_{ij}\right) \frac{d^3}{dt^3}\left(Q^{ij}_{(orb)} + Q^{ij}\right) \right\rangle$$

$$= \frac{1}{5} \left\langle \frac{d^3}{dt^3}\left(Q_{ij(orb)}\right) \frac{d^3}{dt^3}\left(Q^{ij}_{(orb)}\right) \right\rangle + \frac{2}{5} \left\langle \frac{d^3}{dt^3}\left(Q_{ij(orb)}\right) \frac{d^3}{dt^3}\left(Q^{ij}\right) \right\rangle$$

$$+ \frac{1}{5} \left\langle \frac{d^3}{dt^3}\left(Q_{ij}\right) \frac{d^3}{dt^3}\left(Q^{ij}\right) \right\rangle$$

$$P = \frac{1}{5}\left(4\mu r^2 \Omega^3\right)^2 2 + \frac{2}{5}\left(4\mu r^2\Omega^3\right)\left(\frac{\lambda\, m_{BH}}{r^3}\left(12\Omega^3\right)\right) 2 + \mathcal{O}(\lambda^2)$$

Now substituting the value of $r(\Omega)$ using equation ④,

$$P = \frac{1}{5}\left(4\mu\Omega^3\right)^2\left[\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta r)\right]^4 2$$

$$+ \frac{4}{5}\left(4\mu\Omega^3\right)\left(\lambda m_{BH}\, 12\Omega^3\right)\frac{1}{\left[\left(\frac{M}{\Omega^2}\right)^{1/3}(1+\delta r)\right]} + \mathcal{O}(\lambda^2)$$

$$= \frac{2}{5}\left(4\mu\Omega^3\right)^2\left(\frac{M}{\Omega^2}\right)^{4/3}\left(1 + 12\frac{\lambda m_{BH}^2}{\mu M}\left(\frac{\Omega^2}{M}\right)^{5/3}\right)$$

$$+ \frac{4}{5}\left(4\mu\Omega^3\right)\left(\lambda m_{BH}\, 12\Omega^3\right)\left(\frac{\Omega^2}{M}\right)^{1/3} + \mathcal{O}(\lambda^2)$$

$$= \frac{32}{5}\left(\frac{\mu^2}{M^2}\right) M^{10/3}\Omega^{10/3}\left[1 + 12\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}^2}{M^2}\right)\left(\frac{M}{\mu}\right) M^{10/3}\Omega^{10/3}\right.$$
$$\left. + 6\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)\left(\frac{M}{\mu}\right) M^{10/3}\Omega^{10/3} + \mathcal{O}(\lambda^2)\right]$$

$$\boxed{P = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 (M\Omega)^{10/3}\left[1 + 6\left(\frac{\lambda}{M^5}\right)\left(\frac{M}{\mu}\right)\left(\frac{m_{BH}}{M}\right)(M\Omega)^{10/3}\left(1 + 2\frac{m_{BH}}{M}\right) + \mathcal{O}(\lambda^2)\right]} \quad ⑥$$

(e) Gravitational wave phasing:

For this we need, the total energy of the system given by equation (5),

$$E = -\frac{1}{2}\mu M^{2/3}\Omega^{2/3} + \frac{9}{2}\lambda m_{BH}^2\left(\frac{\Omega^2}{M}\right)^2 + \mathcal{O}(\lambda^2)$$

$$\Rightarrow \frac{dE}{d\Omega} = -\frac{1}{3}\mu M^{2/3}\Omega^{-1/3} + 18\lambda m_{BH}^2\frac{\Omega^3}{M^2} + \mathcal{O}(\lambda^2)$$

$$= -\frac{1}{3}\left(\frac{\mu}{M}\right)M^2(M\Omega)^{-1/3} + 18\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)^2 M^2(M\Omega)^3 + \mathcal{O}(\lambda^2)$$

Now computing the phasing using above and equation (6),

$$\frac{d^2\Psi_{SPA}}{d\Omega^2} = 2\,\frac{dE/d\Omega}{\dot{E}}$$

$$= \frac{2\left(\frac{1}{3}\left(\frac{\mu}{M}\right)M^2(M\Omega)^{-1/3} - 18\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)^2 M^2(M\Omega)^3 + \mathcal{O}(\lambda^2)\right)}{\frac{32}{5}\left(\frac{\mu}{M}\right)^2(M\Omega)^{10/3}\left[1+6\left(\frac{\lambda}{M^5}\right)\left(\frac{M}{\mu}\right)\left(\frac{m_{BH}}{M}\right)(M\Omega)^{10/3}\left(1+2\frac{m_{BH}}{M}\right)+\mathcal{O}(\lambda^2)\right]}$$

$$= \frac{10}{96}\left(\frac{M}{\mu}\right)M^2(M\Omega)^{-11/3} - \frac{36\times5}{32}\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)^2\left(\frac{M}{\mu}\right)^2 M^2(M\Omega)^{-1/3}$$

$$- \frac{60}{96}\left(\frac{M}{\mu}\right)^2 M^2(M\Omega)^{-1/3}\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)\left(1+2\frac{m_{BH}}{M}\right) + \mathcal{O}(\lambda^2)$$

$$= \frac{5}{48}\left(\frac{M}{\mu}\right)M^2(M\Omega)^{-11/3} - \left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)M^2\left(\frac{M}{\mu}\right)^2(M\Omega)^{-1/3}\left(\frac{5}{8}+\frac{55}{8}\frac{m_{BH}}{M}\right) + \mathcal{O}(\lambda^2)$$

$$\boxed{\frac{d^2\Psi_{SPA}}{d\Omega^2} = \left(\frac{5}{48}\right)\frac{M^2}{\eta}(x)^{-11/2} - \frac{5}{8}\left(\frac{\lambda}{M^5}\right)\left(\frac{m_{BH}}{M}\right)\frac{M^2}{\eta^2}(x)^{-1/2}\left(1+11\frac{m_{BH}}{M}\right) + \mathcal{O}(\lambda^2)} \quad (7)$$

where, we have used $x = (M\Omega)^{2/3}$ and $\eta = \frac{\mu}{M}$.

From above we can see that the tidal phase correction scales as $x^{-1/2}$ where as leading order term scales as $x^{-11/2}$, which is x^5 smaller than the subleading correction.

II. Black hole Quasi-Normal Modes

The plot of the results for the real and imaginary parts of the $\ell = 2$ quasinormal mode frequencies for a nonspinning black hole for different overtones looks as shown in Fig. 1.

When interpreting $\mathrm{Re}(\omega_{n\ell})$ as an oscillation frequency and $\mathrm{Im}(\omega_{n\ell})$ as a decay rate, the features exhibited in this plot seem counterintuitive based on expectations for the oscillation modes of a string or an elastic body, for which both the oscillation frequency and the decay rate increase with increasing overtone number n, i.e. with an increasing number of nodes in the wavefunction. The QNM plot, however, shows that $\mathrm{Re}(\omega_{n\ell})$ is first decreasing with n, then has a zero, and then increases to an asymptotically constant value for large n.

This behavior can seem more natural when considering a re-interpretation of $\mathrm{Re}(\omega_{n\ell})$ and $\mathrm{Im}(\omega_{n\ell})$. To this end, we consider a simple damped oscillator with amplitude $\psi(t)$, oscillation frequency ω_0, and linear damping γ_0, obeying the equation of motion

$$\ddot{\psi} + \gamma_0 \dot{\psi} + \omega_0^2 \psi = 0. \tag{1}$$

The general solution is of the form

$$\psi(t) = a_1 e^{i\omega_+ t} + a_2 e^{i\omega_- t}, \tag{2}$$

where a_1 and a_2 are constants determined by the initial conditions and

$$\omega_\pm = \pm\sqrt{\omega_0^2 - (\gamma_0/2)^2} + i\frac{\gamma_0}{2}. \tag{3}$$

We see that the solutions are of the form $\exp[(\pm i\omega_{\mathrm{R}} ? \omega_{\mathrm{I}})t]$, with

$$\omega_{\mathrm{R}} = \sqrt{\omega_0^2 - (\gamma_0/2)^2}, \qquad \omega_{\mathrm{I}} = \frac{\gamma_0}{2}. \tag{4}$$

Inverting this to solve for the parameters of the oscillator ω_0 and γ_0 in terms of the oscillation modes of the solution leads to

$$\omega_0 = \sqrt{\omega_{\mathrm{R}}^2 + \omega_{\mathrm{I}}^2}, \qquad \gamma_0 = 2\omega_{\mathrm{I}}. \tag{5}$$

Note that only in the limit $\gamma_0/2 \ll \omega_0$ corresponding to very long-lived modes we get the identification $\omega_0 \approx \omega_{\mathrm{R}}$. However, when modeling the quasinormal modes of black holes arising from oscillator degrees of freedom analogous to those in Eq. (1), the opposite limit applies. This is seen in Fig. 1, where for most modes $\omega_{\mathrm{I}} \gg \omega_{\mathrm{R}}$. In this limit, the frequency of the oscillator degree of freedom is $\omega_0 \approx \omega_{\mathrm{I}}$.

Taking into account the identification (5) between the frequency and damping of the oscillators and the real and imaginary parts of the frequency of the solution leads to the version of Fig. 1 shown in Fig. 2.

We observe that in terms of ω_0 the structure of the black hole frequency spectrum becomes similar to expectations for generic oscillators. The frequency ω_0 increases monotonically with the overtone number n, and since the damping coefficient $\gamma_0 = 2\omega_{\mathrm{I}}$, the damping also increases monotonically with n. Thus, in terms of the equivalent harmonic oscillators, the least damped mode ($n = 1$) also has the lowest value of ω_0, and with increasing ω_0 the lifetime of the excitation becomes shorter.

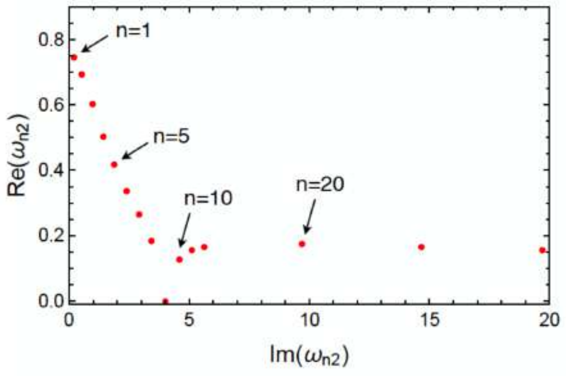


FIG. 1: Real vs. imaginary parts of the $\ell = 2$ quasinormal mode frequencies for a Schwarzschild black hole for different overtones (red dots).

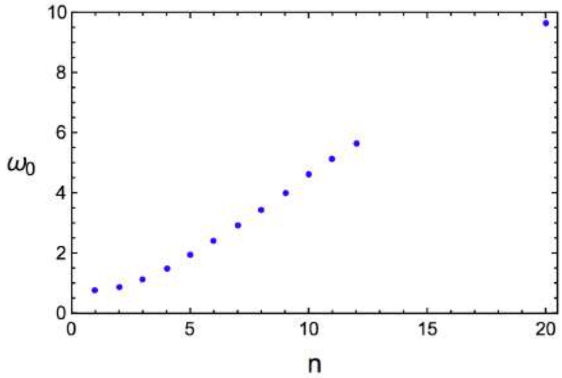


FIG. 2: Frequency of the equivalent oscillator degrees of freedom for quasinormal modes of a Schwazschild black hole (blue dots) computed from the relation (5).

3. **Analytic representation for the quasi-normal modes**

We consider the equation:

$$r(r-1)\psi_{l,rr} + \psi_{l,r} - \left[\frac{\rho^2 r^3}{r-1} + l(l+1) - \frac{s^2-1}{r}\right]\psi_l = 0. \tag{1}$$

a) The singular values of Eq. (1) are $r = 0$ and $r = 1$.

Now, we consider the ansatz:

$$\psi_l = \exp[-\rho(r + \ln r)]. \tag{2}$$

By taking derivatives of this ansatz with respect to r, and substituting the results into Eq. (1), we get:

$$\frac{e^{-\rho(r+\ln r)}}{(r-1)r}\left\{(1+\rho^2) - s^2 + r[s^2 - 1 - 2\rho + l(l+1)] - r^2[l(l+1) + 2\rho^2]\right\} \tag{3}$$

Employing the assumption that $\Re(\rho) > 0$ (so that the exponential always decays), this equation is equal to zero when $r \to \infty$.

Now, we consider the ansatz

$$\psi_l = (r-1)^\rho = \epsilon^\rho. \tag{4}$$

As in the first case, we substitute this ansatz into Eq. (1), and we get the expression:

$$-\frac{(r-1)^\rho}{r}\left\{1 - s^2 + r^2\rho^2 + r^3\rho^2 + r[\rho + l(l+1)]\right\}. \tag{5}$$

If we take the limit $r \to 1$, then this expression is identical to zero, since we have $\Re(\rho) > 0$.

b) and c) These are solved in the attached jupyter notebook.