Gravitational-Wave Course Homework Sheet 7

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Course webpage: https://imprs-gw-lectures.aei.mpg.de/2024-gravitational-waves/

Homework due date: Homeworks must be emailed by Monday, January 6, 2025 to the corresponding Tutor for this homework.
Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with a maximum possible score of **30 points**. Each sub-exercise yields a maximum amount of points as indicated *[in brackets]*.

RECOMMENDED READINGS:

1. Post-Newtonian approximation and effective field theory: M. Levi, Rept. Prog. Phys. **83**, 075901 (2020) [arXiv:1807.01699].
2. Fokker action: T. Damour, G. Esposito-Farese, Phys. Rev. D **53** 5541–5578 (1996) [arXiv:gr-qc/9506063].

EXERCISE:

1. *[30 points]* **Leading-order spin-orbit interaction**

   We consider the action of two *spinning* point-particles labeled $a = 1, 2$ interacting gravitationally to order c^{-2}:

$$
\begin{aligned}
S = {} & \frac{1}{32\pi G}\int dt d^3x \left(4\phi\Delta\phi - \frac{1}{c^2}A_i\Delta A_i\right) \\
& + \sum_{a=1,2}\int dt \left[-m_a\phi + \frac{1}{c^2}\left(\frac{3}{2}v_a^i S_a^{ij}\partial_j\phi + m_a v_a^i A_i + \frac{1}{2}S_a^{ij}\partial_i A_j\right)\right]_{\mathbf{x}=\mathbf{x}_a} \\
& - \frac{1}{8\pi G c^2}\int dt d^3x\, \phi\partial_t^2\phi + \sum_{a=1,2}\int dt\, m_a\left[-c^2 + \frac{\mathbf{v}_a^2}{2} + \frac{1}{c^2}\left(\frac{\mathbf{v}_a^4}{8} - \frac{3\mathbf{v}_a^2}{2}\phi - \frac{1}{2}\phi^2\right)\right]_{\mathbf{x}=\mathbf{x}_a} + \mathcal{O}(c^{-4}).
\end{aligned}
\tag{1}
$$

   Here the spin tensor is given by $S_a^{ij}(t) = -S_a^{ji}(t) = \epsilon_{ijk}S_a^k(t)$ and the spin vector fulfills the usual angular-momentum Poisson bracket $\{S_a^i, S_a^j\} = \epsilon_{ijk}S_a^k$ (a corresponding kinematic term in the action encoding this Poisson bracket is omitted for simplicity). As in the lectures, let us eliminate the gravitational fields ϕ and A_i from the action, keeping contributions up to order c^{-2}, but now pick out the leading-order spin-orbit interactions, that is, terms that are linear in $\mathbf{S}_1$ and independent of $\mathbf{S}_2$:

   - *[10 points]* The last line in (1) does not contribute to the desired result. Briefly argue why, or annotate in the following steps where those terms would enter and show that they either contribute to higher orders in c^{-2} or to the nonspinning case.
   - *[10 points]* Derive the field equations for ϕ and A_i from the action.

- *[10 points]* Eliminate the fields by inserting the solution to the field equations into the action, keeping only terms to the desired order in spins and c^{-2}. As in the lectures, you can make use of the formula $\Delta^{-1}\delta(\mathbf{x}-\mathbf{x}_a) = -1/(4\pi|\mathbf{x}-\mathbf{x}_a|)$ to get an explicit result for the two-body Lagrangian, and drop singular self-interactions. (The result is a so-called Fokker action.)
- *[5 points]* **Optional:** From this Lagrangian, obtain the relative-motion Hamiltonian in the center-of-mass system where $\mathbf{p}_1 = -\mathbf{p}_2$, with definitions given in the previous exercise (the spin variables remain untouched in the Legendre transform).

The analogous result linear in $\mathbf{S}_2$ and independent of $\mathbf{S}_1$ can be obtained by swapping particle labels. **Optional:** *[5 points]* Include in the above calculation the leading-order S_1-S_2 interaction, that is, terms containing both $\mathbf{S}_1$ and $\mathbf{S}_2$.

Note that the spin of black holes can be written as $S_a = Gm_a^2\chi_a/c$ where the dimensionless spin χ_a is less than 1. Hence each order in spin is suppressed by another power of c^{-1}. The leading-order spin-orbit interaction for compact binaries is then at order c^{-3} or 1.5PN, and the S_1-S_2 interaction at order c^{-4} or 2PN.