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1. **Gravitational waves from pulsars:**

(a) *Power emitted in GWs:*

A set of coordinates $\mathbf{x}'$ rotating with the object is related to an inertial coordinate system $\mathbf{x}$ with common origin at the star's center of mass by a rotation matrix

$$x'_i = R_{ij}x^j, \qquad R_{ij} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\phi = \Omega t$ and Ω is the constant rotation frequency. The components of the inertia tensor in the inertial coordinates are therefore obtained by the transformation

$$I_{ij} = R_{ik} I'_{kl} R_{jl}, \tag{2}$$

where $I' = \text{diag}(I_1, I_2, I_3)$. Explicitly,

$$I_{xx} = I_1(\cos\phi)^2 + I_2\sin(\phi)^2 = \frac{1}{2}(I_1 - I_2)\cos(2\phi) + \text{const}, \tag{3}$$

$$I_{yy} = I_1(\sin\phi)^2 + I_2\cos(\phi)^2 = \frac{1}{2}(I_1 - I_2)\cos(2\phi) + \text{const}, \tag{4}$$

$$I_{xy} = I_{yx} = (I_1 - I_2)\sin\phi\cos\phi = \frac{1}{2}(I_1 - I_2)\sin(2\phi) \tag{5}$$

$$I_{zz} = \text{const}, \qquad I_{xz} = I_{yz} = 0. \tag{6}$$

Since $\text{Tr}I' = \text{Tr}I = I_1 + I_2 + I_3 = \text{const}$ we can use (6) directly in place of the quadrupole moment in the quadrupole formula for the energy loss:

$$\begin{aligned} \frac{dE_{\text{GW}}}{dt} &= -\frac{1}{5}\frac{G}{c^5}\langle \dddot{I}^2_{xx} + \dddot{I}^2_{yy} + 2\,\dddot{I}^2_{xy}\rangle && (7) \\ &= -\frac{1}{5}\frac{G}{c^5}\frac{1}{4}(2\Omega)^6(I_1 - I_2)^2\langle(\cos 2\phi)^2 + (\cos 2\phi)^2 + 2(\sin 2\phi)^2\rangle && (8) \\ &= -\frac{32}{5}\frac{G}{c^5}(I_1 - I_2)^2\Omega^6 && (9) \end{aligned}$$

Defining the ellipticity $\epsilon = (I_1 - I_2)/I_3$ we obtain

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G}{c^5}\epsilon^2 I_3^2\Omega^6. \tag{10}$$

(b) *Spindown due to GW emission*

We use the energy balance equation $\dot{E}_{\text{rot}} = -\dot{E}_{\text{GW}}$ with $E_{\text{rot}} = I\Omega^2/2$ for a uniform sphere to obtain

$$\dot{\Omega} = \frac{32}{5}\frac{G}{c^5}\epsilon^2 I\Omega^5 \tag{11}$$

Substituting the values for the Crab pulsar we find that

$$\frac{\dot{\Omega}}{\Omega} \approx 2 \times 10^{-19}\frac{1}{s}. \tag{12}$$

Over an observation time of $\sim$ 3yr$\sim 10^8$s the change in the frequency due to GW losses is very small and the signal remains nearly monochromatic.

(c) *Upper limit on the ellipticity*

Solving Eq. (11) and $\dot{\Omega}/\Omega = -\dot{P}/P$ for ϵ, using $\Omega = 2\pi/(0.033s)$ and assuming that the pulsar has $M = 1.4M_\odot$, $R = 10$km we find that

$$\epsilon \lesssim 7.4 \times 10^{-4}. \tag{13}$$

In reality, the mass, radius, and moment of inertia of the Crab pulsar are uncertain and could differ from the fiducial values given above, which changes the upper limit on ϵ.

The braking index for GW emission is $n = 5$ which is much higher than the observed values for the Crab and Vela pulsars. Pulsars also spin-down due to electromagnetic emission through magnetic dipole radiation, for example, the Crab pulsar radiates a huge amount of power $\sim 10^5 L_\odot$ that is absorbed by and powers the Crab nebula. The small braking index of the Vela pulsar cannot be attributed entirely to radiation from a constant magnetic dipole but might be due to a changing, magnetic moment or effective moment of inertia.

2. **Central-force problem at 1PN order**

A detailed discussion of this problem can be found in §106 of L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields: Volume 2.* We want to recall the 1PN-Lagrangian (or Einstein-Infeld-Hoffman Lagrangian):

$$\mathcal{L} = \mathcal{L}_{\rm N} + \mathcal{L}_{\rm 1PN} + \mathcal{O}(c^{-4}), \tag{14a}$$

$$\mathcal{L}_{\rm N} = \frac{m_1}{2}\vec{v}_1^2 + \frac{m_2}{2}\vec{v}_2^2 + \frac{Gm_1m_2}{r}, \tag{14b}$$

$$\begin{aligned}\mathcal{L}_{\rm 1PN} = &\frac{1}{8c^2}m_1\vec{v}_1^4 + \frac{1}{8c^2}m_2\vec{v}_2^4 - \frac{G^2m_1m_2(m_1+m_2)}{2c^2r^2}\\ &+\frac{Gm_1m_2}{c^2r}\left(\frac{3}{2}\vec{v}_1^2 + \frac{3}{2}\vec{v}_2^2 - \frac{7}{2}\vec{v}_1\cdot\vec{v}_2 - \frac{1}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right),\end{aligned} \tag{14c}$$

in the coordinates $\vec{r}_1 \equiv \vec{x}_1$, $\vec{r}_2 \equiv \vec{x}_2$ and velocities $\vec{v}_1$, $\vec{v}_2$, where $r = |\vec{r}_1 - \vec{r}_2|$, $\vec{n} = (\vec{r}_1 - \vec{r}_2)/r$.

(a) *Canonical momenta*

$$\begin{aligned}&\vec{p}_1 = \frac{\partial\mathcal{L}}{\partial\vec{v}_1} = m_1\vec{v}_1 + \frac{1}{c^2}\left\{\frac{m_1}{2}\left(\vec{v}_1\right)^2\vec{v}_1 + \frac{G_{m_1m_2}}{r}\left[6\vec{v}_1 - 7\vec{v}_2 - \vec{n}\left(\vec{v}_2\cdot\vec{n}\right)\right]\right\} + \mathcal{O}\left(\frac{1}{c^3}\right)\\ &\vec{p}_2 = \vec{p}_1(1\leftrightarrow 2)\end{aligned} \tag{15}$$

Let $\vec{r}_1 = \frac{1}{2}(\vec{R}-\vec{r})$ and $\vec{r}_2 = \frac{1}{2}(\vec{R}+\vec{r})$ where $\vec{R} = (\vec{r}_1+\vec{r}_2)$, as well as $\vec{p}_1 = (\vec{P}-\vec{p})$ and $\vec{p}_2 = (\vec{P}+\vec{p})$ where $\vec{p} = \frac{(\vec{p}_2-\vec{p}_1)}{2}$ and $P = (\vec{p}_1+\vec{p}_2)$.

Now, using the Euler-Lagrange equations

$$\begin{aligned}\frac{d\vec{P}}{dt} &= \frac{d}{dt}\left(\vec{p}_1+\vec{p}_2\right) = \frac{\partial\mathcal{L}}{\partial\vec{r}_1} + \frac{\partial\mathcal{L}}{\partial\vec{r}_2} = \left(\frac{\partial\mathcal{L}}{\partial\vec{R}}\frac{\partial\vec{R}}{\partial\vec{r}_1} + \frac{\partial\mathcal{L}}{\partial\vec{r}}\frac{\partial\vec{r}}{\partial\vec{r}_1}\right) + (1\leftrightarrow 2)\\ &= \left(\frac{\partial\mathcal{L}}{\partial\vec{R}} + \frac{\partial\mathcal{L}}{\partial\vec{r}}\right) + \left(\frac{\mathcal{L}}{\partial\vec{R}} - \frac{\partial\mathcal{L}}{\partial\vec{r}}\right) = 2\frac{\partial\mathcal{L}}{\partial\vec{R}} = 0\end{aligned} \tag{16}$$

Hence $\vec{P}$ is conserved.

(b) *Relative-motion Hamiltonian at 1PN*

Inverting the above equations for $\vec{p}_1$ and $\vec{p}_2$ to get $\vec{v}_1(\vec{p}_1, \vec{p}_2)$ and $\vec{v}_2(\vec{p}_1, \vec{p}_2)$, we can use the trick that $\vec{v}_1 = \vec{p}_1/m_1 + \mathcal{O}(1/c^2)$ to replace v_2 in p_1 and v_1 in p_2. We find

$$\begin{aligned} \vec{v}_1 &= \frac{\vec{p}_1}{m} + \frac{1}{c^2}\left\{-\frac{(\vec{p}_1^2)}{2m_1^3}\vec{p}_1 + \frac{G}{2r}\left[-6\frac{m_2}{m_1}\vec{p}_1 + 7\vec{p}_2 + (\vec{p}_2\cdot\vec{n})\,\vec{n}\right]\right\} + \mathcal{O}\left(\frac{1}{c^3}\right) \\ \vec{v}_2 &= \vec{v}_1(1 \Leftrightarrow 2) \end{aligned} \tag{17}$$

Substituting the above in $\mathcal{L}$, and computing the Hamiltonian using the Legendre transform

$$H = \vec{p}_1\cdot\vec{v}_1 + \vec{p}_2\cdot\vec{v}_2 - \mathcal{L} \tag{18}$$

Also going to center of mass coordinates $\vec{p}_1 = -\vec{p}_2 = -\vec{p}$ and using variables, $\nu = \frac{m_1 m_2}{M^2}$ where $M = (m_1 + m_2)$, we find

$$\begin{aligned} H &= H_0 + \frac{1}{c^2}H_2 + \mathcal{O}\left(\frac{1}{c^3}\right) \\ H_0 &= \frac{\vec{p}^2}{2M\nu} - \frac{GM^2\nu}{r} \\ H_2 &= \left(\frac{3\nu - 1}{8M^3\nu^3}\right)\vec{p}^4 - \frac{G}{2r\nu}\left[(3+\nu)\vec{p}^2 + \nu(\vec{p}\cdot\vec{n})^2\right] + \frac{G^2M^3\nu}{2r^2} \end{aligned} \tag{19}$$

(c) *Binding energy E and angular momentum L*

For circular orbits we have $\dot{r} = 0, (\vec{p}\cdot\vec{n}) = 0$. Assuming the motion in equatorial plane in (r, ϕ, θ) coordinates $\Rightarrow \vec{p} = \frac{L}{r}\hat{\phi}$, where the angular momentum is $\vec{L} = L(\vec{n}\times\hat{\phi})$. Then,

$$E = H\Big|_{\text{circ}} = \frac{L^2}{2r^2M\nu} - \frac{GM^2\nu}{r} + \frac{1}{c^2}\left\{\frac{(3\nu-1)L^4}{8r^4M^3\nu^3} - \frac{(3+\nu)GL^2}{2r^3\nu} + \frac{G^2M^2\nu}{2r^2}\right\} \tag{20}$$

Using Hamiltons equations, we find $\dot{r} = \frac{\partial H}{\partial p_r} = 0$, and $\dot{p}_r = -\frac{\partial H}{\partial r} = 0$, as we are studying stable circular orbits. Furthermore

$$\Omega \equiv \dot{\phi} = \frac{\partial H}{\partial L}, \quad \dot{L} = -\frac{\partial H}{\partial \phi} = 0, \tag{21}$$

and the angular momentum is indeed conserved.

We obtain a relation between L and r using $\frac{\partial H(L,r)}{\partial r} = 0$ given by,

$$L(r) = M\nu\sqrt{GMr}\left[1 + \frac{1}{c^2}\frac{2GM}{r}\right], \tag{22}$$

so we now have $H(r)$ and eliminated L.

Now we obtain a relation between Ω and r using $\frac{\partial H}{\partial L} = \Omega$

$$\Omega = \sqrt{\frac{GM}{r^3}}\left[1 + \frac{1}{c^2}\frac{GM(\nu-3)}{2r}\right] \tag{23}$$

Now inverting the above equation and substituting in $H(r)$,

$$E(\Omega) \equiv H(\Omega) = -\frac{M\nu}{2}v^2\left[1 - \left(\frac{v}{c}\right)^2\left(\frac{\nu+9}{12}\right)\right] \text{ where } v \equiv (M\Omega)^{1/3} \tag{24}$$

(d) *Periastron advance*

We start afresh from $\mathcal{L} = \mathcal{L}_0 + \frac{1}{c^2}\mathcal{L}_2$ with

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\mu\vec{v}^2 + \frac{G\mu M}{r} \\ \mathcal{L}_2 &= \frac{1}{8}\mu(1-3\nu)\vec{v}^4 + \frac{G\mu M}{2r}\left[(3+\nu)\vec{v}^2 + \nu(\vec{v}\cdot\vec{n})^2 - \frac{GM}{r}\right] \end{aligned} \tag{25}$$

but writing above in terms of velocities using

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\vec{v}} = \mu\vec{v} + \frac{1}{c^2}\left\{\frac{1}{2}\mu(1-3\nu)\vec{v}^2\vec{v} + \frac{G\mu M}{r}[(3+\nu)\vec{v} + \nu\vec{n}(\vec{v}\cdot\vec{n})]\right\} \tag{26}$$

give us,

$$\begin{aligned} E &= \vec{p}\cdot\vec{v} - \mathcal{L} \\ &= \frac{1}{2}\mu\vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{3}{8}\mu(1-3\nu)\vec{v}^4 + \frac{G\mu M}{2r}\left[(3+\nu)\vec{v}^2 + \nu(\vec{n}\cdot\vec{v})^2 + \frac{GM}{r}\right]\right\} \end{aligned} \tag{27}$$

Now we use $\vec{v} = \dot{r}\vec{n} + r\dot{\phi}\hat{\phi} \Rightarrow \vec{v}^2 = \dot{r}^2 + r^2\dot{\phi}^2$ where we use $L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ to obtain a relation between $\dot{\phi}$ and L as,

$$\dot{\phi} = \frac{L}{\mu r^2} + \frac{1}{c^2}\left\{\frac{-1}{\mu r^2}\left[\frac{(1-3v)}{2}\left(\dot{r}^2 + \frac{L^2}{\mu^2 r^2}\right)L + \frac{GM(3+\nu)L}{r}\right]\right\} \tag{28}$$

Substituting above in E,

$$\begin{aligned} \frac{E}{\mu} = \frac{\dot{r}^2}{2} - \frac{GM}{r} + \frac{L^2}{2M^2r^2\nu^2} + \frac{1}{c^2}\Bigg\{&\frac{(3-9\nu)}{8}\dot{r}^4 + \left[\frac{3GM}{2r} + \frac{L^2(1-3r)}{4M^2r^2\nu^2} + \frac{GM\nu}{r}\right]\dot{r}^2 \\ &+ \frac{G^2M^2}{2r^2} - \frac{GL^2(3+\nu)}{2Mr^3\nu^2} + \frac{L^4(3\nu-1)}{8M^4r^4\nu^4}\Bigg\} \end{aligned} \tag{29}$$

Now Ignoring $\dot{r}^4$ terms as $\dot{r}$ is expected to be small. Solving for $\dot{r}^2$,

$$\begin{aligned} \dot{r}^2 =& 2\left[\frac{E}{\mu} + \frac{GM}{r} - \frac{L^2}{2M^2r^2\nu^2}\right] + \frac{1}{c^2}\Bigg\{-\frac{2GM}{r}\frac{E}{\mu}(3+2\nu) + \frac{GL^2}{M\nu^3r^3}(5+6\nu) \\ &+ \frac{3L^4}{4M^4\nu^4r^4}(1-3\nu) + \frac{1}{r^2}\left[\frac{L^2}{M^2\nu^2}\frac{E}{\mu}(3\nu-1) - G^2M^2(7+4\nu)\right]\Bigg\} \end{aligned} \tag{30}$$

The above has the structure $\frac{1}{2}\dot{r}^2 + V(r) = \text{constant}$, where $\Omega_r^2 = \left.\frac{\partial^2 V}{\partial r^2}\right|_{\text{circ}}$, which is just the radial frequency. Here we evolve all derivatives in the circular orbit limit hence,

$$
\begin{aligned}
\frac{E}{\mu} &= -\frac{v^2}{2}\left[1 - \frac{v^2}{c^2}\left(\frac{9+\nu}{12}\right)\right] \quad r = \frac{GM}{v^2}\left[1 + \frac{v^2}{c^2}\left(\frac{\nu}{3} - 1\right)\right] \\
L &= \frac{\nu M^2 G}{v}\left[1 + \frac{v^2}{c^2}\frac{(9+\nu)}{6}\right] \\
\Rightarrow \quad \Omega_r^2 &= \frac{v^6}{G^2 M^2}\left[1 - \frac{6v^2}{c^2}\right] = \Omega^2\left[1 - \frac{6v^2}{c^2}\right]
\end{aligned}
\tag{31}
$$

Now fractional advance of periastron is given by,

$$
\frac{\Delta\Phi}{2\pi} = k(\Omega) - 1 = \frac{1}{\sqrt{1 - \frac{6v^2}{c^2}}} - 1 = \frac{3v^2}{c^2}
\tag{32}
$$

(e) *The stability of circular orbits*

In polar coordinates (with $G = 1$ and $\mu = 1$ for simplicity),

$$
\frac{H}{\mu} = \frac{\vec{p}^2}{2} - \frac{M}{R} + \frac{1}{c^2}\left\{\frac{(3\nu - 1)}{8}\vec{p}^4 - \frac{M}{R}\left[\frac{(3+\nu)}{2}\vec{p}^2 + \frac{\nu\vec{p}_r^2}{2}\right] + \frac{M^2}{2R^2}\right\}.
\tag{33}
$$

We can now use Hamilton's eqs of motion:

$$
\begin{aligned}
\dot{R} &= \frac{\partial H}{\partial R}, \quad \dot{P}_R = -\frac{\partial H}{\partial R} \\
\Omega &= \frac{\partial H}{\partial \phi}, \quad \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0
\end{aligned}
\tag{34}
$$

For circular orbit $R = R_0, P_r = 0, \dot{P}_r = 0$. And we let

$$
\begin{aligned}
R &= R_0 + \delta R, \quad P_R = \delta P_R \\
\Omega &= \Omega_0 + \delta\Omega, \quad P_\phi = P_{\phi_0} + \delta P_\phi
\end{aligned}
\tag{35}
$$

at linear order.

$$
\begin{aligned}
\dot{R} + \delta\dot{R} &= \frac{\partial}{\partial P_r}\left[H\left(R_0 + \delta R, \delta P_R, P_{\phi_0} + \delta P_\phi\right)\right] \\
&= \underbrace{\frac{\partial}{\partial P_R}\left[H\left(R_0, 0, P_{\phi_0}\right)\right]}_{=\dot{R}} + \underbrace{\frac{\partial^2}{\partial P_R \partial R}\left[H\left(R_0, 0, P_{\phi_0}\right)\right]}_{\frac{\partial H}{\partial R}\big|_{R=R_0, P_R=0} = 0}\delta R \\
&+ \underbrace{\frac{\partial^2}{\partial P_R^2}\left[H\left(R, 0, P_{\phi_0}\right)\right]}_{\equiv C_0}\delta P_R + \underbrace{\frac{\partial^2}{\partial P_R \partial P_\phi}\left[H\left(R, 0, P_{\phi_0}\right)\right]}_{\frac{\partial H}{\partial P_R}\big|_{R=R_0, P_R=0} = 0}\delta P_\phi
\end{aligned}
\tag{36}
$$

Which implies

$$
\delta\dot{R} = C_0\ \delta P_R
\tag{37}
$$

As $\dot{P}_\phi = 0$, one finds with a similar procedure $\delta\dot{P}_\phi = 0$ and

$$\delta\dot{P}_R = -A_0\delta R - B_0\delta P_\phi, \quad \delta\Omega = B_0\delta R + D_0\delta P_\phi \tag{38}$$

where

$$\begin{aligned}
A_0 &= \left[\frac{3P_{\phi_0}^2}{R_0^4} - \frac{2M}{R^3} + \frac{1}{c^2}\left\{\frac{3M^2}{R_0^4} - 6M(3+\nu)\frac{P_{\phi_0}^2}{R_0^5} + \frac{5}{2}(3\nu-1)\frac{P_{\phi_0}^4}{R_0^6}\right\}\right] \\
B_0 &= \left[\frac{-2P_{\phi_0}}{R_0^3} + \frac{1}{c^2}\left\{3M(3+\nu)\frac{P_{\phi_0}}{R_0^4} - 2(3\nu-1)\frac{P_{\phi_0}^3}{R_0^5}\right\}\right] \\
C0 &= \left[1 + \frac{1}{c^2}\left\{\frac{(3\nu-1)}{2}\left(\frac{P_{\phi_0}}{R_0}\right)^2 - (3+2\nu)\frac{M}{R_0}\right\}\right] \\
D_0 &= \left[\frac{1}{R_0^2} + \frac{1}{c^2}\left\{-\frac{M}{R_0^3}(3+\nu) + \frac{3}{2}(3v-1)\frac{P_{\phi_0}^2}{R_0^4}\right\}\right]
\end{aligned} \tag{39}$$

Now we look for solution of the form $\sim e^{i\sigma t}$

$$\delta P_R = \frac{\delta\dot{R}}{C_0} \Rightarrow \delta\dot{P}_R = \frac{\delta\ddot{R}}{C_0} = -A_0\delta R - B_0\delta P_\phi \tag{40}$$

Now $\delta\dot{P}_\phi = 0 \Rightarrow \delta P_\phi =$ constant (which we set to zero) and

$$\delta\ddot{R} = -A_0C_0\delta R \;\Rightarrow\; \sigma = \pm\sqrt{A_0C_0} \tag{41}$$

Now to have a stable (oscillatory) solution, if $\Sigma \equiv A_0C_0 > 0$. If we compute the above within the circular limit we find

$$R_0 = \frac{M}{v^2}\left[1 + \frac{v^2}{c^2}\left(\frac{v}{3} - 1\right)\right], \quad P_\phi = \frac{M}{v}\left[1 + \frac{v^2}{c^2}\frac{\nu+9}{6}\right] \tag{42}$$

which implies

$$\Sigma = \Omega^2\left(1 - 6\frac{v^2}{c^2}\right) \tag{43}$$

Notice that the ISCO is reached when $\Sigma = 0 \Rightarrow \frac{v^2}{c^2} = \frac{1}{6} \Rightarrow r = 6GM$. This is however an accident, which does not hold at high PN orders!

Now recall, $\Omega_r^2 = \Omega^2\left[1 - \frac{6v^2}{c^2}\right]$, hence $\Omega_r = 0$ coincides with $\Sigma = 0$, which means that perturbations of ISCO have ∞ orbital period i.e. ISCO is an unstable orbit.