Gravitational-Wave Course Homework Sheet 6

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Course webpage: `https://imprs-gw-lectures.aei.mpg.de/2024-gravitational-waves/`

Homework due date: Homework must be emailed by Monday, December 2, 2024 to the corresponding Tutor for this homework.
Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with a maximum possible score of **30 points**. Each sub-exercise yields a maximum amount of points as indicated *[in brackets]*.

RECOMMENDED READINGS:

1. E. E. Flanagan and T. Hinderer, Phys. Rev. D **77**, 021502 (2008) [arXiv:0709.1915]
2. M. Maggiore, Phys. Rev. Lett. **100**, 141301 (2008)
3. B. F. Schutz and C. M. Will, Astrophys.J.Lett. **291** (1985), L33-L36
4. V. Ferrari and B.Mashhoon, Phys. Rev. **D30** (1984), 295
5. E. Berti (2004), `https://arxiv.org/abs/gr-qc/0411025`
6. E. W. Leaver, Proc. R. Soc. Lond. A 402, 285-298 (1985), `https://www.edleaver.com/Misc/EdLeaver/Publications/AnalyticRepresentationForQuasinormalModesOfKerrBlackHoles.pdf`

EXERCISES:

1. **Newtonian quadrupolar tidal imprint in the GW phasing** *[15 points]*

   Consider a neutron star-black hole binary system of total mass M and reduced mass μ whose orbital motion is described by Newtonian gravity. The Lagrangian is

   $$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} - \frac{1}{2}Q_{ij}\mathcal{E}_{ij} + L_{\rm int}, \tag{1}$$

   where $L_{\rm int}$ describes the internal dynamics of the quadrupole and the Newtonian tidal field is

   $$\mathcal{E}_{ij} = -m_{\rm BH}\partial_i\partial_j(1/r) = -m_{\rm BH}(3n^in^j - \delta^{ij})/r^3, \tag{2}$$

   where $n^i = x^i/r$ is a unit vector. Note that $n^in_i = 1$ and $\delta^{ij}\delta_{ij} = 3$. Assume that the quadrupole is adiabatically induced and given by

   $$Q^{\rm ad}_{ij} = -\lambda\mathcal{E}_{ij}, \tag{3}$$

   where λ is the tidal deformability parameter. The internal Lagrangian then describes only the elastic potential energy $L^{\rm ad}_{\rm int} = -Q_{ij}Q^{ij}/(4\lambda)$. Throughout this exercise, assume that tidal effects are small and can be treated as linear perturbations.

   (a) *[3 points]* Obtain the equations of motion for r and ϕ from the Euler-Lagrange equations.

(b) *[3 points]* Assume that the orbit is circular ($\ddot{r} = 0$ and $\dot{\phi} = \Omega$). Starting from the radial equation of motion, express the radius as $r(\Omega) = M^{1/3}\Omega^{-2/3}(1 + \delta r)$ and compute the linear tidal corrections δr.

(c) *[3 points]* Calculate the energy of the system from (1). Specialize to adiabatic quadrupoles and circular orbits, and express the energy in terms of Ω.

(d) *[3 points]* The leading order gravitational radiation is generated by the total quadrupole of the system $Q^T_{ij} = Q^{\rm orbit}_{ij} + Q_{ij}$. Compute the tidal contribution to the energy flux from the quadrupole formula.

(e) *[3 points]* In the stationary phase approximation (SPA) for the gravitational wave signal, the phasing can be computed from the formula

$$\frac{d^2\Psi_{\rm SPA}}{d\Omega^2} = 2\frac{dE/d\Omega}{\dot{E}_{\rm GW}}. \tag{4}$$

Compute the tidal contribution to $\Psi_{\rm SPA}$, to linear order in the tidal effects. Express your result in terms of the post-Newtonian parameter $x = (M\Omega)^{2/3} = (\pi M f_{\rm GW})^{2/3}$ and show that the tidal phase correction scales as x^5 relative to the leading order phasing.

2. **Black-hole quasi-normal modes** *[10 points]*

In the lectures it was shown that the quasinormal modes (QNMs) of a Schwarzschild black hole are characterized by complex frequencies $\omega = \omega_R + i\omega_I$, with ω_R and ω_I the real and the imaginary parts, respectively.

(a) *[5 points]* Use Table I from arXiv:gr-qc/0411025 to plot ω_R and ω_I of the quadrupolar mode ($l = 2$) versus n, where n is the overtone number that identifies the number of nodes in the radial wavefunction (plus 1 in the reference's conventions). Use n = 1–12, 20, 30, 40. [Note that the values in Table I correspond to $(\omega_R, -\omega_I)$ in our conventions, given the time-dependence of the QNMs as $e^{i\omega t}$.]

Your plot should exhibit some features which could be considered strange according to certain intuition, interpreting ω_R as an oscillation frequency and ω_I as a decay rate. For typical systems with a set of vibrational modes, like a string or an elastic body, both the oscillation frequency and the decay rate increase with increasing overtone number, i.e. with an increasing number of nodes in the wavefunction. The QNM plot, however, shows that ω_R is first decreasing with n, then has a zero, and then increases to an asymptotically constant value. This behavior can be seen as somewhat less mysterious by reinterpreting ω_R and ω_I as follows.

(b) *[5 points]* Consider a simple damped oscillator with amplitude $\psi(t)$ obeying

$$\ddot{\psi} + \gamma_0\dot{\psi} + \omega_0^2\psi = 0. \tag{5}$$

Writing the two linearly independent solutions as $\exp((\pm i\omega_R - \omega_I)t)$, find the relationship between ω_R, ω_I and ω_0, γ_0. Invert this relation, make plots of ω_0 and γ_0 versus n for the Schwarzschild QNMs and comment how this interpretation alleviates the above discussion.

3. **Analytic representation for the quasi-normal modes** *[5 points]*

In this exercise we want to calculate the gravitational quasi-normal frequencies of Schwarzschild black holes by constructing exact eigensolutions to the radiative boundary-value problem of Chandrasekhar and Detweiler. The method is that employed by Jaffe in his determination of the electronic spectra of hydrogen molecule ion in 1934 as applied to black holes in Leaver's paper from 1985.

For this, we choose Schwarzschild coordinates and let $\psi(t, r, \theta, \phi)$ denote a component of a perturbation to a massless spin s field. After Fourier transforming and expanding in spherical harmonics

$$\psi(t, r, \theta, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \left(\sum_l \frac{1}{r} \psi_l(r, \omega) Y_{lm}(\theta, \phi) \right) , \tag{6}$$

it suffices to write the resulting ordinary differential equation satisfied by $\psi_l(r, \omega)$ in the form, where t and r are scaled such that $c = G = 2M = 1$. We will additionally introduce $\rho = -i\omega$ to make the computation simpler to find

$$r(r-1)\psi_{l,rr} + \psi_{l,r} - \left[\frac{\rho^2 r^3}{r-1} + l(l+1) - \frac{s^2 - 1}{r} \right] \psi_l = 0. \tag{7}$$

This differential equation belongs to the class of *generalized spheroidal wave equations.*

(a) *[5 points]* We want to study the differential equation Eq. (7). What are the singular values of this equation? We want to solve the ODE in between those values. Show therefore, that $\psi_l \to \exp[-\rho(r + \ln r)]$ solves the differential equation in the limit $r \to \infty$. Equivalently, show that $\psi_l \to (r-1)^\rho$ is a solution as $r \to 1$.
Hint: You may assume that $\rho = \mathrm{Re}(\rho) + i\,\mathrm{Im}(\rho)$ has real part larger than zero and that therefore $\exp[-\rho(r + \ln r)] \to 0$ for $r \to \infty$.

As we are only interested in in-going radiation at the horizon ($r = 1$) and outgoing radiation to infinity, we find the boundary conditions

$$\psi_l \xrightarrow{r \to 1} (r-1)^\rho \text{ and } \psi_l \xrightarrow{r \to \infty} r^{-\rho} e^{-\rho r}. \tag{8}$$

An ansatz for the solution of Eq. (7) is then

$$\psi_l = (r-1)^\rho r^{-2\rho} e^{-\rho(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r} \right)^n . \tag{9}$$

The sequence of expansion coefficients $\{a_n : n = 1, 2...\}$ is determined by a three-term recurrence relation with the recurrence coefficients

$$\alpha_n = n^2 + (2\rho + 2)n + 2\rho + 1 \tag{10}$$

$$\beta_n = -[2n^2 + (8\rho + 2)n + 8\rho^2 + 4\rho + l(l+1) - s^2 + 1] \tag{11}$$

$$\gamma_n = n^2 + 4\rho n + 4\rho^2 - s^2 \tag{12}$$

where it can be shown (see Leaver's paper) that the quasi-normal modes $\rho = -i\omega$ are precisely the solutions to the implicit characteristic equation

$$0 = \beta_0 - \cfrac{\alpha_0 \gamma_1}{\beta_1 - \cfrac{\alpha_1 \gamma_2}{\beta_2 - \cfrac{\alpha_2 \gamma_3}{\beta_3 - \dots}}} \tag{13}$$

as only these ρ values lead to absolute convergence of Eq. (9).

(b) [**Optional!**, *5 points*] Using either Mathematica or Python, code up a function of the form

```
def continued_fraction(omega,N):
    rho = -1j*omega
    CF = beta(rho,N)
    for n in range(N)[::-1]:
        CF = beta(rho,n) - (alpha(rho,n)*gamma(rho,n+1)/CF)
    return CF
```

which computes the continued fraction of Eq. (13). The Schwarzschild quasi-normal modes are precisely the values of ω for which

```
continued_fraction(omega,N) = 0.
```

Solve this for the fundamental mode with the help of `findroot` from the `mpmath` package in python or `FindRoot` in Mathematica.

A good starting estimate for the fundamental mode is $\omega_0 = 0.8 - 0.2i$. Check your result against the tabulated value from the previous exercise. How many recursive steps N do you need for a stable solution?

(c) [**Optional!**, *5 points*] Try to replicate the plot from Exercise I. with this method.