Gravitational-Wave Course Homework Sheet 4

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Course webpage: `https://imprs-gw-lectures.aei.mpg.de/2024-gravitational-waves/`

Homework due date: Homework must be emailed by Monday, November 11, 2024 to the corresponding Tutor for this homework.
Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with a maximum possible score of **30 points**. Each sub-exercise yields a maximum amount of points as indicated *[in brackets]*.

RECOMMENDED READINGS:

1. A. Buonanno and T. Damour, Phys. Rev. **D59** (1999) 084006.
2. A. Buonanno and T. Damour, Phys. Rev. **D62** (2000) 064015.
3. M. Khalil, A. Buonanno, J. Steinhoff, and J. Vines, Phys. Rev. **D104** (2021), 024046.

EXERCISES:

1. **On the effective-one-body Hamiltonian and dynamics** *[15 points]*

   We have derived in class the mapping between the *real* PN Hamiltonian and the *effective* Hamiltonian using the Hamilton-Jacobi formalism. Here we want to construct the effective-one-body (EOB) Hamiltonian using a canonical transformation.

   Using reduced (or dimensionless) variables $\mathbf{Q}, \mathbf{P}$ and $\hat{H}_{\text{eff}}$, the effective Hamiltonian reads

$$\hat{H}_{\text{eff}}(Q,P) = c^2 \sqrt{A(Q)\,\left[1 + \frac{1}{c^2}\,\mathbf{P}^2 + \left(\frac{A(Q)}{D(Q)} - 1\right)\,\frac{1}{c^2}\,(\mathbf{N}\cdot\mathbf{P})^2\right]}\,, \tag{1}$$

   where $\mathbf{N} = \mathbf{Q}/Q$ and

$$A(Q) \;=\; 1 + \frac{a_1}{c^2\,Q} + \frac{a_2}{c^4\,Q^2} + \frac{a_3}{c^6\,Q^3} + \cdots\,, \tag{2}$$

$$D(Q) \;=\; 1 + \frac{d_1}{c^2\,Q} + \frac{d_2}{c^4\,Q^2} + \cdots\,, \tag{3}$$

   where a_i, d_i are unknown coefficients that will be determined by the mapping to the (reduced) PN Hamiltonian

$$\hat{H}_{\text{real}}(q,p) \;=\; \hat{H}_{\text{Newt}}(q,p) + \frac{1}{c^2}\,\hat{H}_{\text{1PN}}(q,p) + \cdots\,, \tag{4}$$

$$\hat{H}_{\text{Newt}}(q,p) \;=\; \frac{1}{2}\mathbf{p}^2 - \frac{1}{q}\,, \tag{5}$$

$$\hat{H}_{\text{1PN}}(q,p) \;=\; -\frac{1}{8}(1-3\nu)\,\mathbf{p}^4 - \frac{1}{2q}\left[(3+\nu)\,\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\right] + \frac{1}{2q^2}\,, \tag{6}$$

where $\mathbf{q}$ and $\mathbf{p}$ are reduced variables, $\mathbf{n} = \mathbf{q}/q$ and $\nu = m_1\, m_2/(m_1 + m_2)^2$, being m_1 and m_2 the black-hole masses. At 1PN order the real and effective Hamiltonians are related as

$$1 + \frac{\hat{H}_{\rm real}(q,p)}{c^2}\left(1 + \alpha_1\,\frac{\hat{H}_{\rm real}(q,p)}{c^2}\right) = \frac{\hat{H}_{\rm eff}(Q(q,p),P(q,p))}{c^2}\,, \tag{7}$$

where α_1 is an unknown coefficient that will be determined by the mapping. The canonical transformation at 1PN order is

$$Q^i \;=\; q^i + \frac{1}{c^2}\frac{\partial G_{\rm 1PN}}{\partial p_i}\,, \tag{8}$$

$$P_i \;=\; p_i - \frac{1}{c^2}\frac{\partial G_{\rm 1PN}}{\partial q^i}\,, \tag{9}$$

with

$$G_{\rm 1PN}(\mathbf{q},\mathbf{p}) = (\mathbf{q}\cdot\mathbf{p})\,\left[c_1\mathbf{p}^2 + \frac{c_2}{q}\right]\,, \tag{10}$$

where c_1, c_2 are unknown coefficients that will be determined by the mapping.

The goal of this exercise is to determine α_1, c_1, c_2 as a function of ν.
Insert the canonical transformation given in Eqs. (8) and (9) in Eq. (7) and expand the latter in PN orders through 1PN order. By equating terms with the same structures in $\mathbf{q}$, $\mathbf{p}$, derive the equations for the unknown coefficients a_1, α_1, c_1, c_2 and set $a_2 = a_3 = ... = a_n = d_1 = d_2 = ... = d_n = 0$. In this case you should find that: $\alpha_1 = \nu/2$, $c_1 = -\nu/2$ and $c_2 = 1 + \nu/2$. [Hint: introduce the parameter $\epsilon^2 \equiv 1/c^2$, work with the square of Eq. (7) to get rid of the square root in Eq. (1), and neglect the terms with order higher than $O(\epsilon^4)$. Note that it is sufficient to derive $Q \equiv |\mathbf{Q}| = \sqrt{Q^i\, Q_i}$, $P \equiv |\mathbf{P}| = \sqrt{P^i\, P_i}$ and $\mathbf{N}\cdot\mathbf{P} = N^i\, P_i$ as function of $q \equiv |\mathbf{q}|$, $p \equiv |\mathbf{p}|$ and $\mathbf{n}\cdot\mathbf{p}$ through 1PN order using the canonical transformation given in Eqs. (8) and (9).]

2. **Incorporating the emission of gravitational waves in the two-body dynamics** *[15 points]*

The Hamiltonian of the gravitational two-body problem allows us to determine the *conservative* (no loss of energy) two-body dynamics through Hamilton equations of motion (EOMs). However, gravitational waves (GWs) carry away energy and angular momentum from the binary system, so Hamilton EOMs must be extended to account for these *dissipative* effects. We can achieve this through the use of *balance equations*, which relate the GW fluxes of energy Φ_E and angular momentum Φ_L, to the losses of energy dE/dt and angular momentum dL/dt of the system, respectively.

More specifically, if we choose a coordinate system in the effective one-body frame such that the z-axis is aligned with the orbital angular momentum of the system $L = p_\phi$ (and hence the z-axis is perpendicular to the orbital plane of the binary system), we can write the extended Hamilton EOMs for a Hamiltonian H as

$$\dot{r} = \frac{\partial H}{\partial p_r}, \tag{11a}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi}, \tag{11b}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} + \mathcal{F}_r, \tag{11c}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} + \mathcal{F}_\phi, \tag{11d}$$

where r is the separation of the binary, ϕ is the azimuthal angle, p_r is the radial momentum, and p_ϕ is the angular momentum. In these equations, we have added a radiation-reaction (RR) force $\mathcal{F}$ (with radial and azimuthal components $\mathcal{F}_r$ and $\mathcal{F}_\phi$, respectively) acting on the binary whose purpose is to account for the energy and angular momentum losses due to the emission of GWs. For the rest of the exercise, we assume azimuthal symmetry so the term $\partial H/\partial\phi$ can be ignored.

The goal of this exercise is to understand how $\mathcal{F}_r$ and $\mathcal{F}_\phi$ are related to the fluxes of energy and angular momentum. In this way, we will be able to model the *dynamics* of two bodies (e.g., two black holes) which inspiral and, eventually, merge. Here, we will cover only the inspiral part (see, e.g., this YouTube video `https://www.youtube.com/watch?v=KwbXxzgA0bU`)

NOTE: For all the calculations in this exercise, you can employ `Mathematica` or similar software to ease the calculations. If this is the case, please share with the corresponding tutor the notebook employed for your calculations.

a) *[1 point]* Show that the system of equations (11) satisfies the *balance laws*:

$$\dot{r}\mathcal{F}_r + \dot{\phi}\mathcal{F}_\phi = -\Phi_E, \tag{12a}$$

$$\mathcal{F}_\phi = -\Phi_L. \tag{12b}$$

This equation is correct for binaries moving on circular orbits, but it is incomplete for binaries moving on eccentric orbits, as we will see in item d).

b) *[8 points]* [Based on Secs. 4.1.2 and 4.1.3 of Maggiore's book, Vol. 1]
We introduce the Keplerian parametrization of the orbit

$$r = \frac{R}{1 + e\cos\psi}, \tag{13}$$

where R is the latus-rectum, e is the eccentricity of the orbit, and ψ is the angular position of the effective one-body particle with reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ (at Newtonian order, and thus throughout this exercise, we have $\psi = \phi$). Employing this parametrization of the orbit, obtain the quadrupole moments Q_{ij} of the source (two point particles moving in a generic, planar orbit), and show that the orbit-averaged (i.e., averaged over one orbit) fluxes of energy and angular momentum are given by:

$$\langle\Phi_E\rangle = \frac{32}{5}\frac{G^4\mu^2 M^3}{c^5 a^5}\frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right), \tag{14a}$$

$$\langle\Phi_L\rangle = \frac{32}{5}\frac{G^{7/2}\mu^2 M^{5/2}}{c^5 a^{7/2}}\frac{1}{(1-e^2)^2}\left(1 + \frac{7}{8}e^2\right). \tag{14b}$$

where $M = m_1 + m_2$ and $a = R/(1-e^2)$.
Hint: To obtain the orbit-averaged fluxes, employ the quadrupole approximation:

$$\langle\Phi_E\rangle = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\right\rangle, \tag{15a}$$

$$\langle\Phi_L\rangle = \frac{2G}{5c^5}\epsilon^{3ij}\left\langle \ddot{Q}_{ik}\dddot{Q}_{jk}\right\rangle, \tag{15b}$$

where ϵ^{3ij} is the Levi-Civita symbol with the first index equal to 3 (this is because Eq. (15b) is giving the angular momentum flux in the z-direction).

c) *[2 points]* For the rest of the exercise, we will employ geometric units in which $G = c = 1$ and use reduced variables to simplify the notation in the expressions. The reduced variables are obtained with the following transformations

$$\frac{t}{M}\to t, \quad \frac{r}{M}\to r, \quad \frac{p_r}{\mu}\to p_r, \quad \frac{p_\phi}{M\mu}\to p_\phi, \quad \frac{E}{\mu}\to E, \quad \frac{\Phi_E}{\nu}\to\Phi_E, \quad \frac{\Phi_L}{M\nu}\to\Phi_L, \tag{16}$$

where $\nu = \mu/M$ is the symmetric mass ratio of the binary. By employing the Newtonian Hamiltonian (in reduced variables)

$$H = \frac{p^2}{2} - \frac{1}{r}, \qquad \text{with} \qquad p^2 = p_r^2 + \frac{p_\phi^2}{r^2}, \tag{17}$$

and the Keplerian parametrization of the orbit (13), obtain expressions for the variables p_r and p_ϕ in terms of e, a, and ψ. For this purpose, assume that there is no dissipation, i.e., neglect the RR force.

Hint: Since there is no dissipation, the energy and angular momentum of the binary are conserved along any point of the orbit. In particular, consider the points $\psi = 0$ and $\psi = \pi$.

d) *[4 points]* The balance laws shown in Eq. (12) are incomplete for eccentric systems. The numerical value of the fluxes on the right-hand-side (RHS) of Eq. (12) is *independent* of the system of coordinates (these are the fluxes measured by an observer at infinity). However, the RR force components in the left-hand-side (LHS) of Eq. (12) *depend* on the system of coordinates. This is because eccentricity is a concept associated with the *deformation* of the orbit, and this deformation depends on the system of reference, according to Einstein's Theory of General Relativity.

To solve this inconsistency between the LHS and the RHS in Eq. (12), one needs to add extra contributions to the balance laws, known as *Schott terms* [1]. These contributions enter as total time derivatives which vanish when they are averaged over one orbit. Therefore, for eccentric orbits, Eq. (12) is correct only when it is orbit-averaged:

$$\langle \dot{r}\mathcal{F}_r + \dot{\phi}\mathcal{F}_\phi \rangle = -\langle \Phi_E \rangle, \tag{18a}$$

$$\langle \mathcal{F}_\phi \rangle = -\langle \Phi_L \rangle. \tag{18b}$$

It can be shown that the coordinate-dependent nature of the RR force can be absorbed in two constants α and β such that the RR force components, at leading Newtonian order and in reduced variables, are given by

$$\mathcal{F}_r = \frac{8\nu}{15\,r^3}\,p_r\left[(-3\alpha + 9\beta + 3)\,p^2 + (9\alpha - 15\beta + 9)\,p_r^2 + \frac{9\alpha - 9\beta + 17}{r}\right], \tag{19a}$$

$$\mathcal{F}_\phi = \frac{8\nu}{15\,r^3}\,p_\phi\left[9(\alpha + 1)\,p_r^2 - 3(2 + \alpha)\,p^2 + \frac{3(\alpha - 2)}{r}\right]. \tag{19b}$$

Show that the RR force components in Eq. (19) satisfy Eq. (18) for any value of α and β.

e) *[Optional, 5 points]* Substitute the Hamiltonian (17) and the RR force (19) into the EOMs (11), and solve numerically this system of ordinary differential equations (ODEs) with e.g. `Mathematica`, `Python`, etc.

Note that this is an ODE system for the four variables (r, ϕ, p_r, p_ϕ). Therefore, we need four initial conditions in order to solve this system. The initial value of ϕ can be set to zero, $\phi_0 = 0$. The rest of the initial values $(r_0, p_{r0}, p_{\phi 0})$ can be determined with Eq. (13) and the relations found in item c). These equations will give us the initial values $(r_0, p_{r0}, p_{\phi 0})$ in terms of the initial values of the eccentricity e and latus-rectum R (set $\psi_0 = \phi_0 = 0$).

For definiteness, consider an equal-mass binary (which value of ν corresponds to an equal-mass binary?), with starting values of $e = 0.3$, $R = 20$, $\psi = 0$, and set $\alpha = -16/3$, $\beta = -13/2$.

Plot the trajectory of the effective particle in the xy-plane, with $x = r\cos\phi$ and $y = r\sin\phi$.

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[1] The details about these Schott terms fall outside the scope of this homework. The interested reader can find some useful information in [M. Khalil, A. Buonanno, J. Steinhoff, and J. Vines, Phys. Rev. **D104** (2021), 024046].