Gravitational-Wave Course — Homework Sheet 3

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Course webpage: https://imprs-gw-lectures.aei.mpg.de/2024-gravitational-waves/

Homework due date: Homeworks must be emailed by Monday, November 4, 2024 to the corresponding Tutor for this homework.

Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with a maximum possible score of **30 points**. Each sub-exercise yields a maximum amount of points as indicated *[in brackets]*.

RECOMMENDED READINGS:

1. Secs. 1 & 5 in R. Abbott et al. https://iopscience.iop.org/article/10.3847/2041-8213/abb655/pdf.
2. There will be a few lectures on post-Newtonian (PN) theory toward the end of the course. For the central-force problem below, it is not needed to know how such Lagrangian was derived. For interested people, a straightforward derivation of the 1PN Lagrangian, originally computed by Einstein-Infeld-Hoffmann in 1938, can be found here: https://fiteoweb.unige.ch/~maggiore/GWVol1/EIHLagrangian.pdf (Dr. Justin Vines corrected the derivation given in Sec. 1.5.5 of Maggiore's book.).

EXERCISES:

I. GRAVITATIONAL WAVES FROM PULSARS [12 POINTS]

Neutron stars possess a rigid crust that is 10 billion times stronger than steel and can support a "mountain" of up to ∼few cm height. Consider a neutron star rotating with angular frequency Ω around a principal body axis $\mathbf{e}_3$ and with constant principal moments of inertia I_1, I_2, I_3. Assume that the neutron star has a deformation such that $I_1 \neq I_2$.

a) *[4 points]* Consider the inertia tensor $I_{ij} = \int d^3x\rho\left(r^2\delta_{ij} - x^i x^j\right)$ that is given by $I_{ij} = \mathrm{diag}(I_1, I_2, I_3)$ in the body frame whose axes rotate with the neutron stars. Compute the components of the inertia tensor in an inertial frame. Use the analogy between I_{ij} and the Newtonian quadrupole moment to obtain the power radiated in gravitational waves. Express your result in terms of the ellipticity ϵ and I_3, where

$$\epsilon = \frac{I_1 - I_2}{I_3} \tag{1}$$

b) *[4 points]* Consider a neutron star that is approximated as a uniform density sphere with mass $\sim 1.4M_\odot$ and $R \sim 10$km so that $I_3 \sim \frac{2}{5}MR^2 \sim 10^{45}\mathrm{g\,cm}^2$. Its rotational energy is $E = I_3\Omega^2/2$. For the Crab pulsar, the rotational period is $P = 33$ms. Use the balance between the energy radiated in gravitational waves and the change in E to obtain its spin-down rate $\dot{\Omega}$. Show that for a fiducial ellipticity of $\epsilon = 10^{-7}$

the rate of change in the frequency is small and thus the GWs are approximately monochromatic over ∼few years observation time.

c) *[4 points]* The observed spindown rate of the Crab pulsar is $\dot{P} = 4.2 \times 10^{-13} s/s$. Assuming that this is caused solely by GW emission, what would the ellipticity of the Crab pulsar need to be to explain this value?

In several pulsars, the spindown rate has been measured with pulsar timing observations and is generally quantified by a braking index n defined by $\dot{\Omega} \propto \Omega^n$. For the Crab pulsar, $n \approx 2.5$ ($n \sim 3$ is expected for magnetic dipole radiation), while for the Vela pulsar $n \approx 1.5$. Read off the braking index from your result (b). Is GW emission the dominant mechanism for the spindown of the Crab pulsar?

II. CENTRAL-FORCE PROBLEM AT 1PN ORDER [18 POINTS]

Starting from the 1PN-Lagrangian (also known as the Einstein-Infeld-Hoffman Lagrangian)

$$\mathcal{L} = \mathcal{L}_{\rm N} + \mathcal{L}_{\rm 1PN} + \mathcal{O}(c^{-4}), \tag{2a}$$

$$\mathcal{L}_{\rm N} = \frac{m_1}{2}\mathbf{v}_1^2 + \frac{m_2}{2}\mathbf{v}_2^2 + \frac{G m_1 m_2}{r}, \tag{2b}$$

$$\begin{aligned}\mathcal{L}_{\rm 1PN} = &\frac{1}{8c^2} m_1 \mathbf{v}_1^4 + \frac{1}{8c^2} m_2 \mathbf{v}_2^4 - \frac{G^2 m_1 m_2 (m_1+m_2)}{2c^2 r^2} \\ &+ \frac{G m_1 m_2}{c^2 r}\left(\frac{3}{2}\mathbf{v}_1^2 + \frac{3}{2}\mathbf{v}_2^2 - \frac{7}{2}\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_1\cdot\mathbf{n}\,\mathbf{v}_2\cdot\mathbf{n}\right),\end{aligned} \tag{2c}$$

in the coordinates $\mathbf{r}_1 \equiv \mathbf{x}_1$, $\mathbf{r}_2 \equiv \mathbf{x}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$, where $r = |\mathbf{r}_1 - \mathbf{r}_2|$, $\mathbf{n} = (\mathbf{r}_1 - \mathbf{r}_2)/r$:

a) *[6 points]* Derive the canonical momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. [Recall from classical mechanics that $\mathbf{p}_a = \partial\mathcal{L}/\partial\mathbf{v}_a$.] Then, introduce the variables $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{P} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, and $\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1)/2$, and show that $\mathbf{P}$ is conserved.

b) *[6 points]* Obtain the relative-motion Hamiltonian $H = \mathbf{p}_1 \cdot \mathbf{v}_1 + \mathbf{p}_2 \cdot \mathbf{v}_2 - \mathcal{L}$ at 1PN order in the variables $\mathbf{r}$, $\mathbf{p}$, $M = m_1 + m_2$ and $\nu = m_1 m_2/M^2$. [Hint: in carrying out the calculation here and below keep only terms at 1PN order! It is also strongly suggested to use Mathematica to manipulate long algebraic expressions.]

c) *[6 points]* Compute the binding energy $E = H$ and orbital angular momentum L at 1PN order for circular orbits. Express the final result for E and L in terms of the velocity $v \equiv (M\Omega)^{1/3}$, where Ω is the orbital frequency. [Hint: Impose the circular orbit condition and derive the relation between r and Ω. You will find a few new terms at 1PN order beyond the usual Newtonian relation $M/r^3 = \Omega^2$. You might find it convenient to work with Hamilton's equations in spherical coordinates and choose the motion to be in the equatorial plane.]

d) Compute the periastron advance at 1PN order for nearly circular orbits. [Hint: It is more convenient to employ the relative-motion Lagrangian. Use the conservation of energy and angular momentum to derive the equation for the radial perturbation around a circular orbit and compute the radial frequency Ω_r as function of Ω. The fractional advance of the periastron per radial period is $\Delta\Phi/(2\pi) = K(\Omega) - 1$, where $K(\Omega) = \Omega/\Omega_r$.] **[optional!]**

e) Study the stability of circular orbits using the 1PN Hamiltonian. **[optional!]**

Consider the polar coordinates (r, ϕ, p_r, p_ϕ) and a perturbation of the circular orbit defined by

$$\begin{aligned} p_r &= \delta p_r\,, \\ p_\phi &= p_\phi^0 + \delta p_\phi\,, \\ r &= r_0 + \delta r\,, \\ \Omega &= \Omega_0 + \delta\Omega\,, \end{aligned}$$

where r_0, Ω_0 and p_ϕ^0 refer to the unperturbed circular orbit. Write down the Hamilton equations and linearize them around the circular orbit solution. You should find

$$\begin{aligned}
\delta\dot{p}_r &= -A_0\,\delta r - B_0\,\delta p_\phi\,,\\
\delta\dot{p}_\phi &= 0\,,\\
\delta\dot{r} &= C_0\,\delta p_r\,,\\
\delta\Omega &= B_0\,\delta r + D_0\,\delta p_\phi\,,
\end{aligned} \tag{3}$$

where A_0, B_0, C_0 and D_0 depend on the unperturbed orbit. Determine explicitly A_0, B_0, C_0 and D_0.

Look at solutions of Eqs. (3) proportional to $e^{i\sigma t}$ and find the criterion of stability. [Hint: you should find that there exists a combination Σ_0 of A_0, B_0, C_0 and D_0 such that when $\Sigma_0 > 0$ the orbits are stable. The innermost stable circular orbit (ISCO) corresponds to $\Sigma_0 = 0$].

Express Σ_0 as function of $v = (M\Omega)^{1/3}$ and show that for any value of the binary mass ratio the ISCO at 1PN order coincides with the Schwarzschild ISCO. [This is an accident, which does not hold at high PN orders!]

Finally, show that $\Sigma_0 = 0$ coincides with $\Omega_r = 0$. What is the physical meaning of this result?