Lecture Recording

- **Note: These lectures will be recorded and posted onto the IMPRS website**

- Dear participants,
- We will record all lectures on "*Making sense of data: introduction to statistics for gravitational wave astronomy*", including possible Q&A after the presentation, and we will make the recordings publicly available on the IMPRS lecture website at:
    - https://imprs-gw-lectures.aei.mpg.de/2023-making-sense-of-data/
- By participating in this Zoom meeting, you are giving your explicit consent to the recording of the lecture and the publication of the recording on the course website.

Making sense of data: introduction to statistics for gravitational wave astronomy

Lecture 4: Examples of frequentist statistics in GW data analysis

AEI IMPRS Lecture Course

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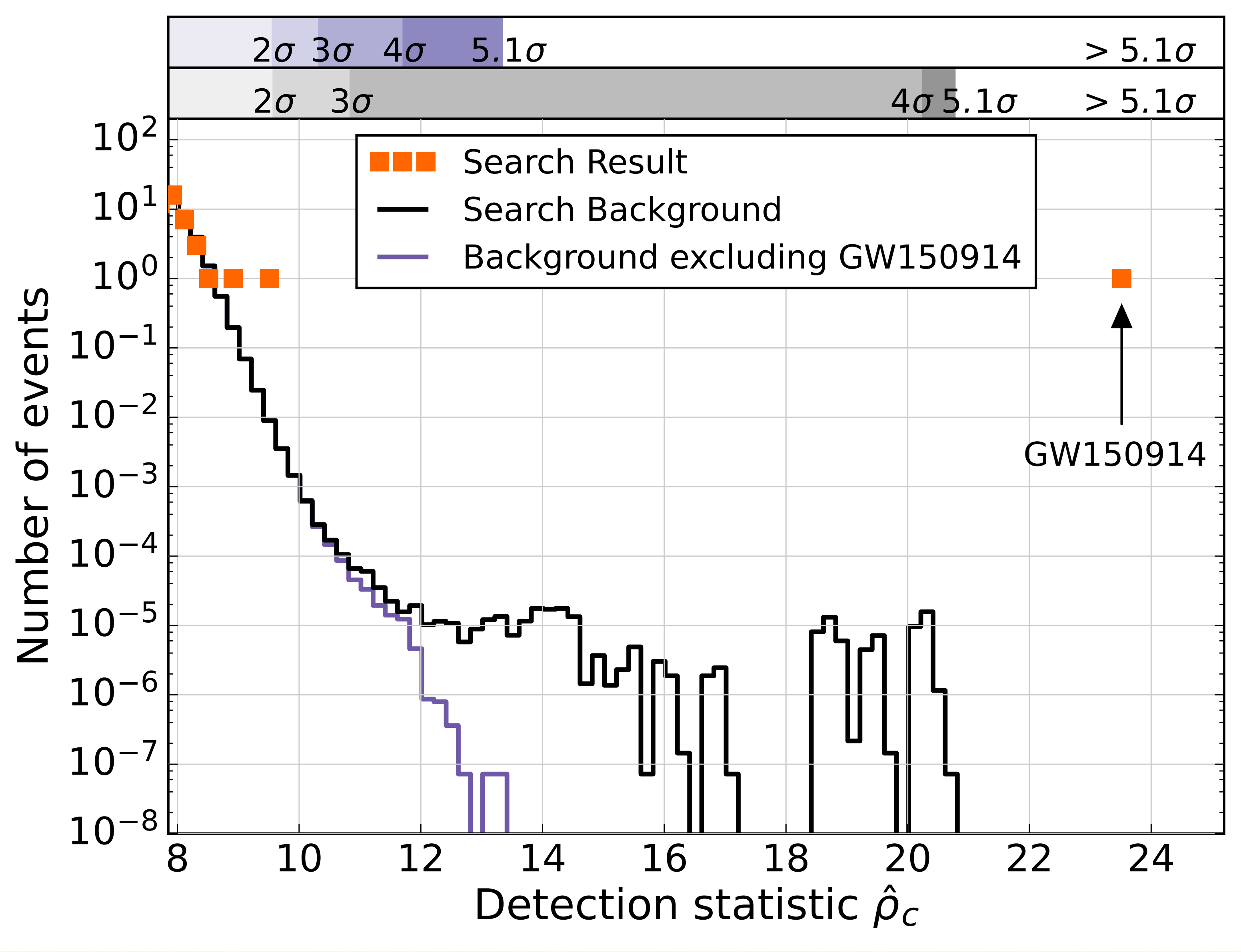


Likelihood

- The usual assumed model for the output of a GW detector is

$$s(t) = n(t) + h(t;\vec{\lambda})$$

- and for stationary noise we have (the meaning of S_n(f) will be discussed further in lecture 5)

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \frac{1}{2}S_n(f)\delta(f-f')$$

- If we additionally assume the noise is Gaussian then we can write down a probability distribution for *s(t)*

$$p(s|\vec{\lambda}) = p(n(t) = s(t) - h(t;\vec{\lambda})) \propto \exp\left[-\frac{1}{2}(s-h(\vec{\lambda})|s-h(\vec{\lambda}))\right]$$

- where

$$(a|b) = \int_{-\infty}^{\infty} \frac{\tilde{a}^*(f)\tilde{b}(f) + \tilde{a}(f)\tilde{b}^*(f)}{S_n(f)}\,\mathrm{d}f$$

Fisher Matrix Estimates of Precision

- Recall the Cramer-Rao bound on the variance of unbiased estimators

$$\mathrm{cov}(\hat{\lambda}_i, \hat{\lambda}_j) \geq \left[\mathbf{\Gamma}_\theta^{-1}\right]_{ij}$$

- where $\hat{\lambda}$ is some estimator of the parameter values and

$$\Gamma_{ij} = \mathbb{E}\left[\frac{\partial l}{\partial \lambda_i}\frac{\partial l}{\partial \lambda_j}\right]$$

- is the Fisher Information Matrix.
- For the gravitational wave likelihood

$$\Gamma_{ij} = \left(\frac{\partial h}{\partial \lambda_i}\middle|\frac{\partial h}{\partial \lambda_j}\right)$$

Linear Signal Approximation

- The Fisher matrix provides a lower bound on the variance, or uncertainty, of an estimator.
- In general, the Fisher matrix provides a good guide to how well parameters can be measured, particularly in the limit of high signal-to-noise ratio. This can be seen in the **linear signal approximation**. If we write

$$s(t) = n(t) + h(t;\vec{\lambda}_0)$$

- and expand

$$\vec{\lambda} = \vec{\lambda}_0 + \Delta\vec{\lambda} \qquad\qquad h(t;\vec{\lambda}) = h(t;\vec{\lambda}_0) + \partial_i h(t;\vec{\lambda}_0)\Delta\lambda^i$$

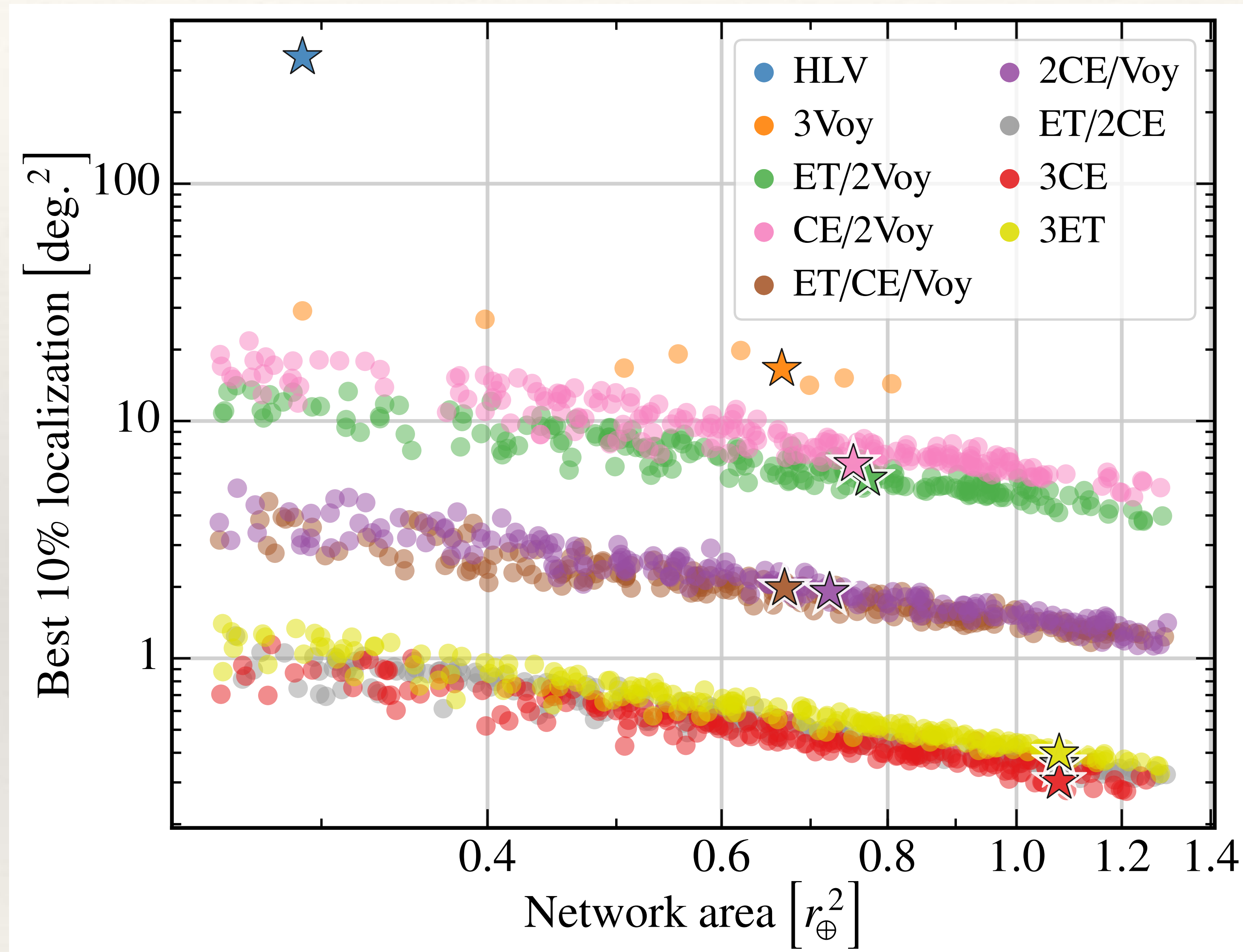
- we find

$$p(s|\lambda) \propto \exp\left[-\frac{1}{2}\left(\Delta\lambda^i - (\Gamma^{-1})_{ik}(n|\partial_k h(t;\lambda_0))\right)\Gamma_{ij}\left(\Delta\lambda^j - (\Gamma^{-1})_{jl}(n|\partial_0 h(t;\lambda_0))\right)\right]$$

- where Γ_{ij} is the Fisher Matrix.

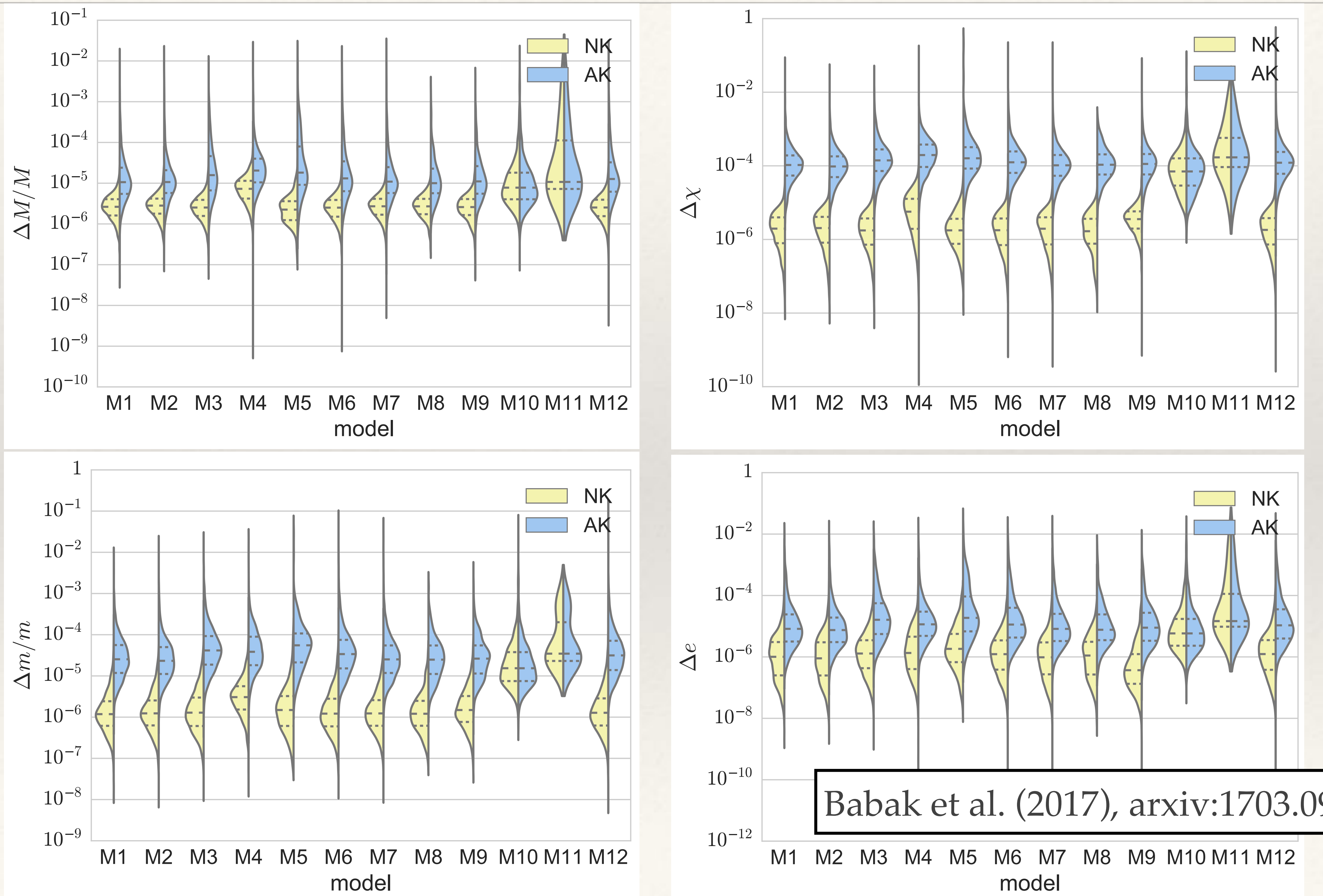
Fisher Matrix Estimates of Precision

- The Fisher matrix is widely used in a gravitational wave context to characterise our ability to measure parameters.
- While only an approximation, it is much cheaper to compute than the posterior or MLE distribution and so can be used to survey parameter space much more efficiently.
- Widely used to scope out science for and optimise design of new facilities.



Hall & Evans (2019), arxiv:1902.09485

Fisher Matrix Estimates of Precision



Babak et al. (2017), arxiv:1703.09722

Matched filtering/template bank searches

Matched Filtering

- The output of a *filter*, the overlap of the data with a template, is a statistic. The *optimal filter* for finding a known signal is one that matches the signal in the Fourier domain, weighted by the noise PSD (see lecture 5)

$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)}$$

- Considering the log-likelihood

$$\begin{aligned} l(\lambda) &= -\frac{1}{2}(\mathbf{s} - A\hat{\mathbf{h}}(\lambda)|\mathbf{s} - A\hat{\mathbf{h}}(\lambda)) = -\frac{1}{2}\left[(\mathbf{s}|\mathbf{s}) - 2A(\mathbf{s}|\hat{\mathbf{h}}) + A^2\right] \\ &= -\frac{1}{2}\left[(\mathbf{s}|\mathbf{s}) + (A - (\mathbf{s}|\hat{\mathbf{h}}))^2 - (\mathbf{s}|\hat{\mathbf{h}})^2\right]. \end{aligned}$$

- We deduce that the maximum of the optimal filter over the parameter space is the **maximum likelihood estimator**.

Matched Filtering

- The optimal filter can also be interpreted in terms of hypothesis testing. Assuming all parameters except the amplitude are known then the question "Is there a gravitational wave in the data" is equivalent to the hypothesis test

$$H_0 : A = 0, \qquad \text{vs.} \qquad H_1 : A > 0$$

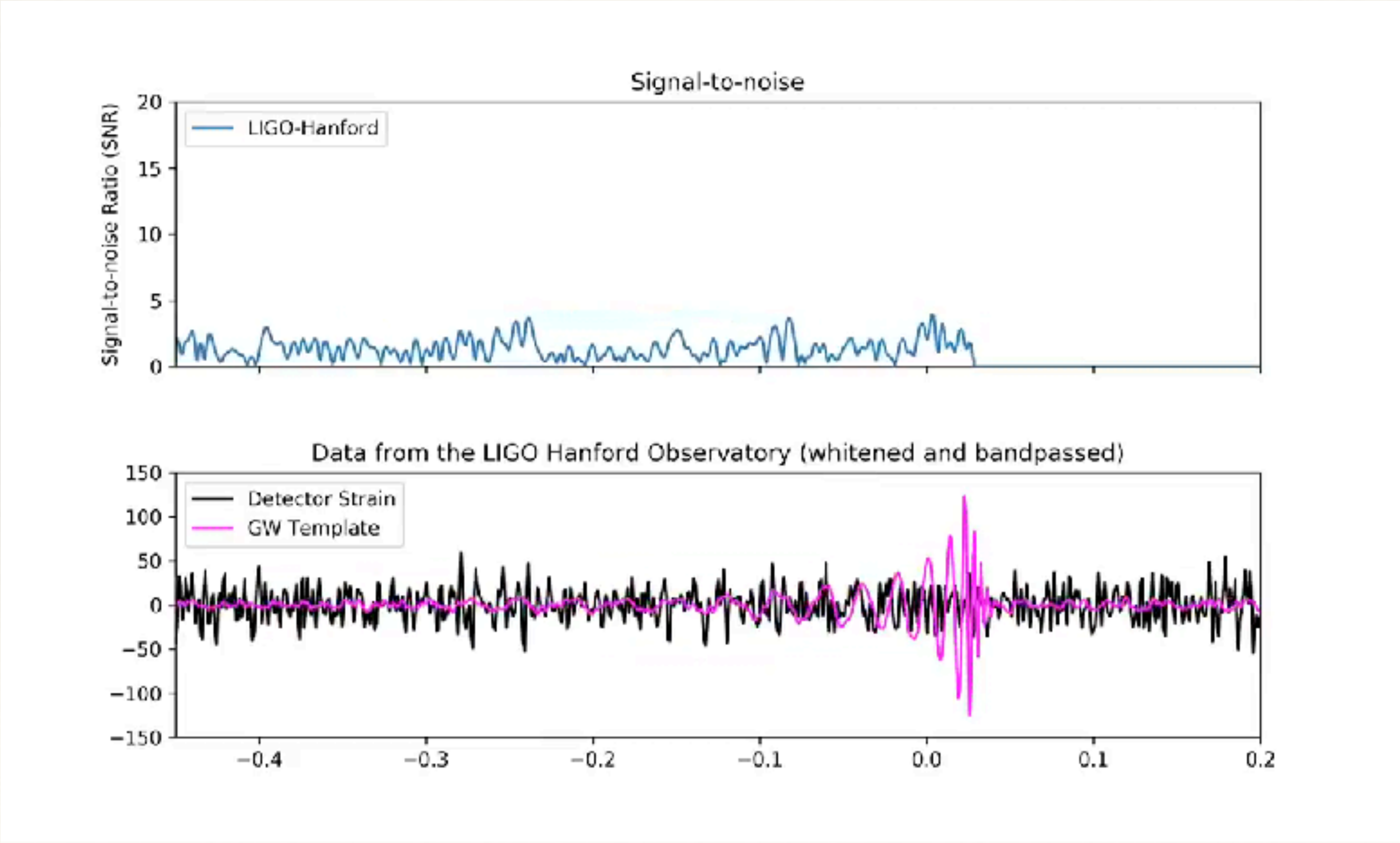
- From the Neyman-Pearson Lemma, the most powerful statistic for testing **A=0** versus $\mathbf{A=A_1 > 0}$ is the likelihood ratio

$$\exp\left[A_1(\mathbf{s}|\hat{\mathbf{h}}(\lambda)) - \frac{1}{2}A_1^2\right]$$

- We deduce that the optimal filter is also the most powerful test statistic for this hypothesis. As the statistic does not depend on A it is also uniformly most powerful for the composite hypothesis **A > 0**.

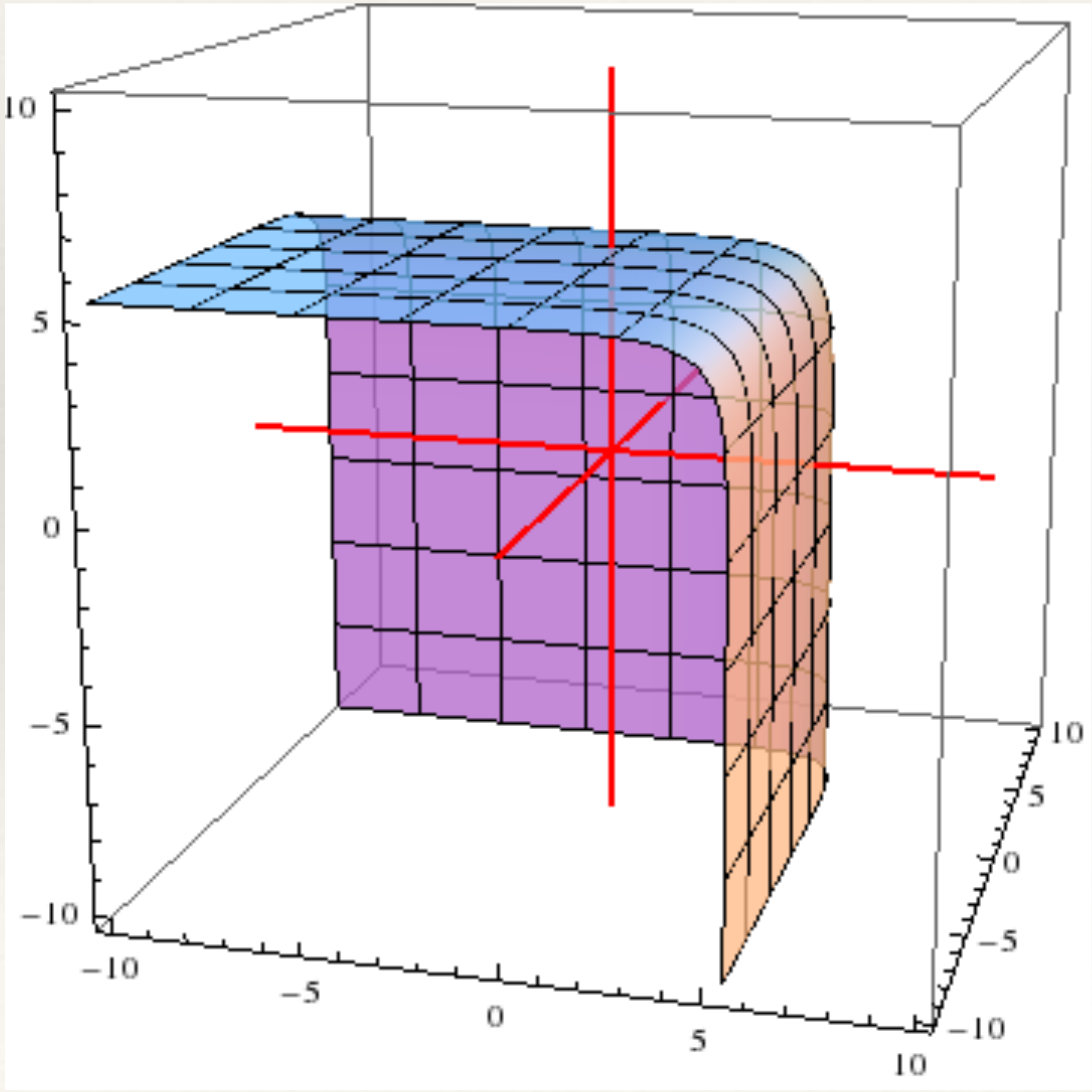
Matched Filtering

- Many searches are based on matched filtering. Since the signal is not known, we employ a *template bank* of possible waveforms. If a signal matches a template in the bank, we can pull it out of the data.



Matched Filtering

- The detection statistic (the maximum overlap in the template bank), is (approximately) the maximum likelihood estimator. It is not the most powerful hypothesis test statistic, but it is a reasonable approximation.

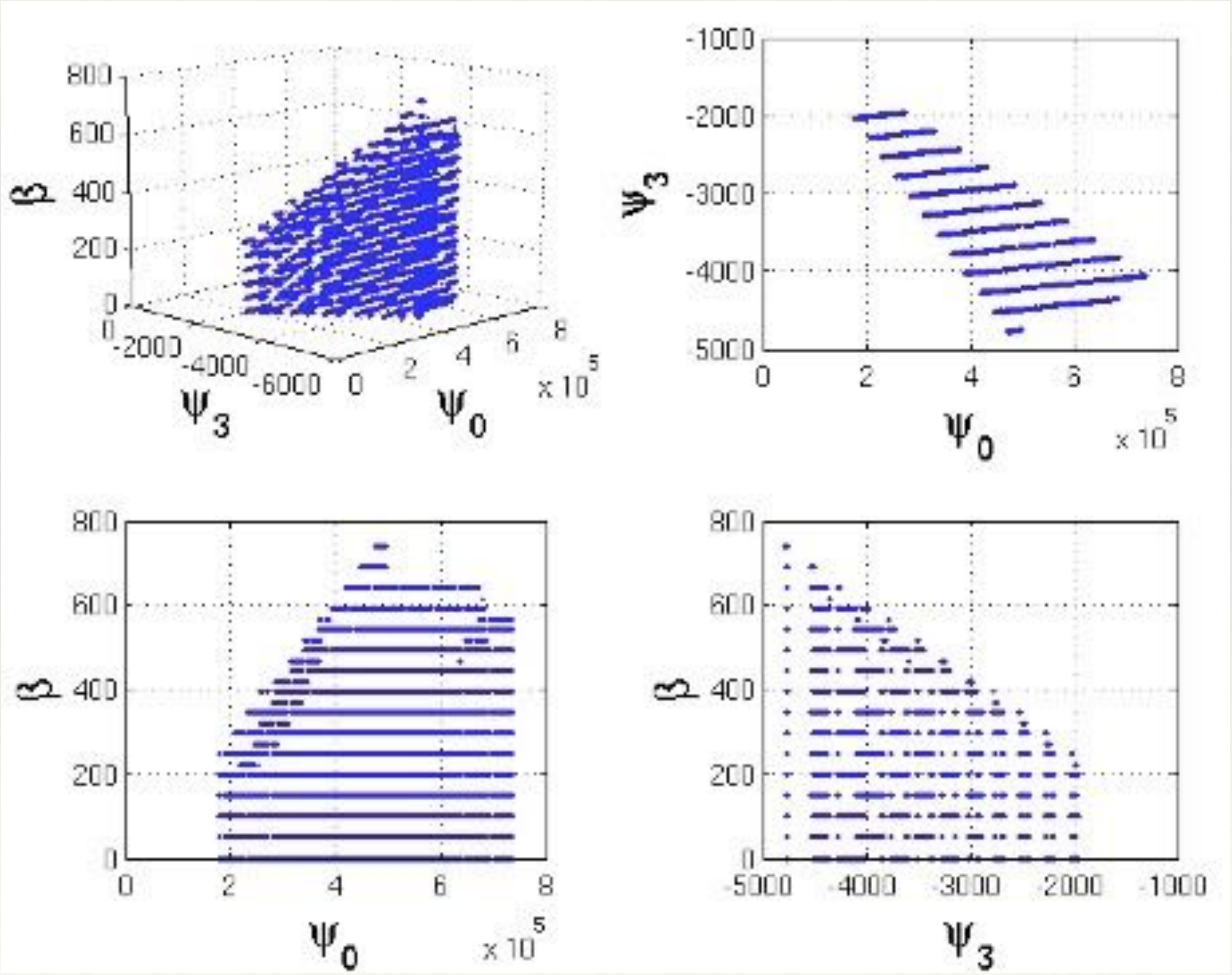


AJK Chua & J Gair (2015)

Template Bank Construction

- The Fisher matrix can also be used as a metric to construct a template bank satisfying a *minimal match* criterion

$$\min_{h_{\rm true}} \max_{h_{\rm temp}} (h_{\rm true}|h_{\rm temp}) \gtrsim 1 - {\rm MM}$$



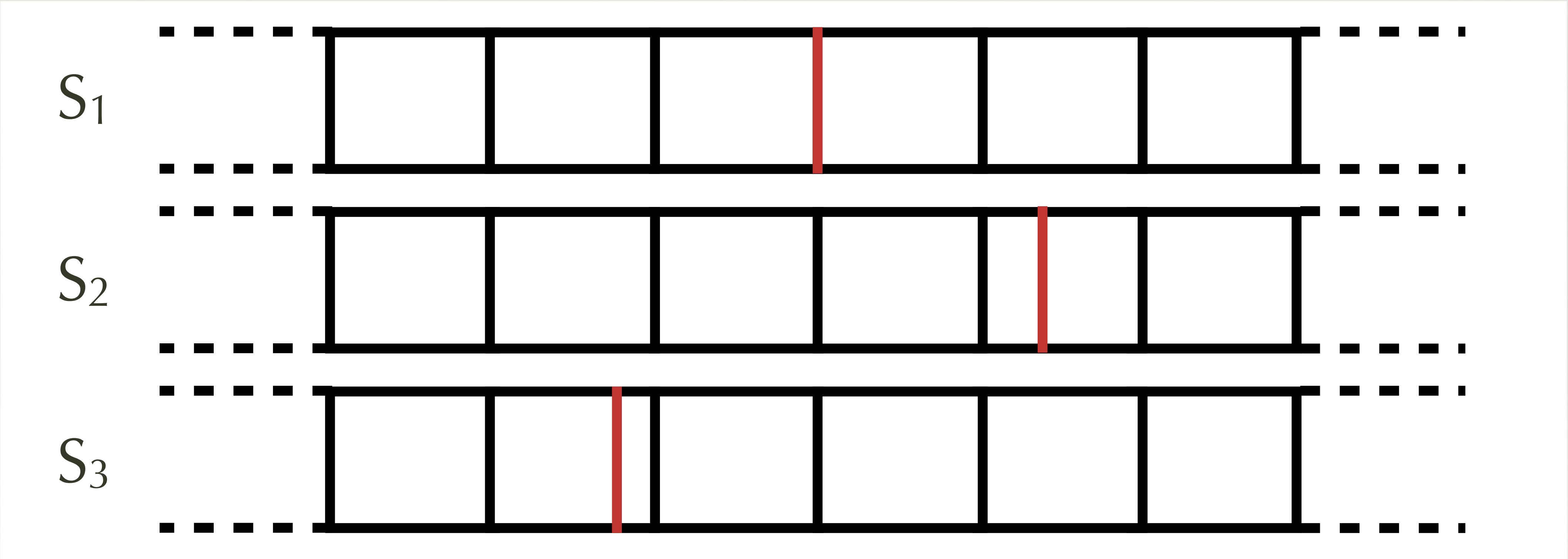
- Fisher Matrix metric not easy to use in higher dimensional parameter spaces. Now common to use *stochastic banks*.
- Can also do *stochastic searches* (MCMC) that generate templates on the fly.

LIGO searches

- Whether using the optimal filter directly, or refined versions of it, LIGO searches are essentially frequentist. There is a defined **detection statistic** and if the detection statistic is exceeded, the data is flagged as interesting.
- The **distribution of the detection statistic** is computed in the absence of signals, and the threshold determined based on the probability of a false detection.
- The **sensitivity** of the search is assessed by injection and recovery of sources into the detector data.
- The **background distribution** is typically evaluated using **time slides**. This approach relies on the fact that LIGO searches normally use data from more than one detector.

Background Estimation

- Noise is not stationary or Gaussian and contains glitches, lines etc.
- Use frequentist techniques to characterise noise background properties
    - process data in a way that eliminates signal but not noise
    - for LIGO - time slide data from different detectors



    - noise + signal coincidences are not background
    - significance of events in tail, i.e., sources, is hard to estimate

False Alarm Rate (FAR)

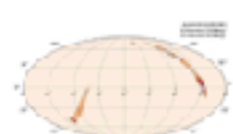
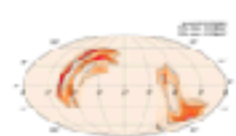
- In the hypothesis testing part of this course we placed a lot of emphasis on the **significance** or **p-value** of a test. These make sense when we are referring to finite data sets. However, in a gravitational wave context, data is continuously being collected. It therefore makes sense to quote a **false alarm rate** (FAR).

- The FAR is the frequency with which triggers of the observed threshold value or higher occur in continuous observation with the detector in the absence of signals.

LIGO/Virgo O3 Public Alerts

Detection candidates: 46

SORT: EVENT ID (A-Z) ▼

Event ID	Possible Source (Probability)	UTC	GCN	Location	FAR	Comments
S200115j	MassGap (>99%)	Jan. 15, 2020 04:23:09 UTC	GCN Circulars Notices | VOE		1 per 1513 years	
S200114f		Jan. 14, 2020 02:08:18 UTC	GCN Circulars Notices | VOE	No public skymap image found.	1 per 25.838 years	
S200112r	BBH (>99%)	Jan. 12, 2020 15:58:38 UTC	GCN Circulars Notices | VOE		1 per 2469.9 years	

https://gracedb.ligo.org

Search refinements: waveform consistency

- If we subtract the correct template the residual at each frequency should be Normally distributed.

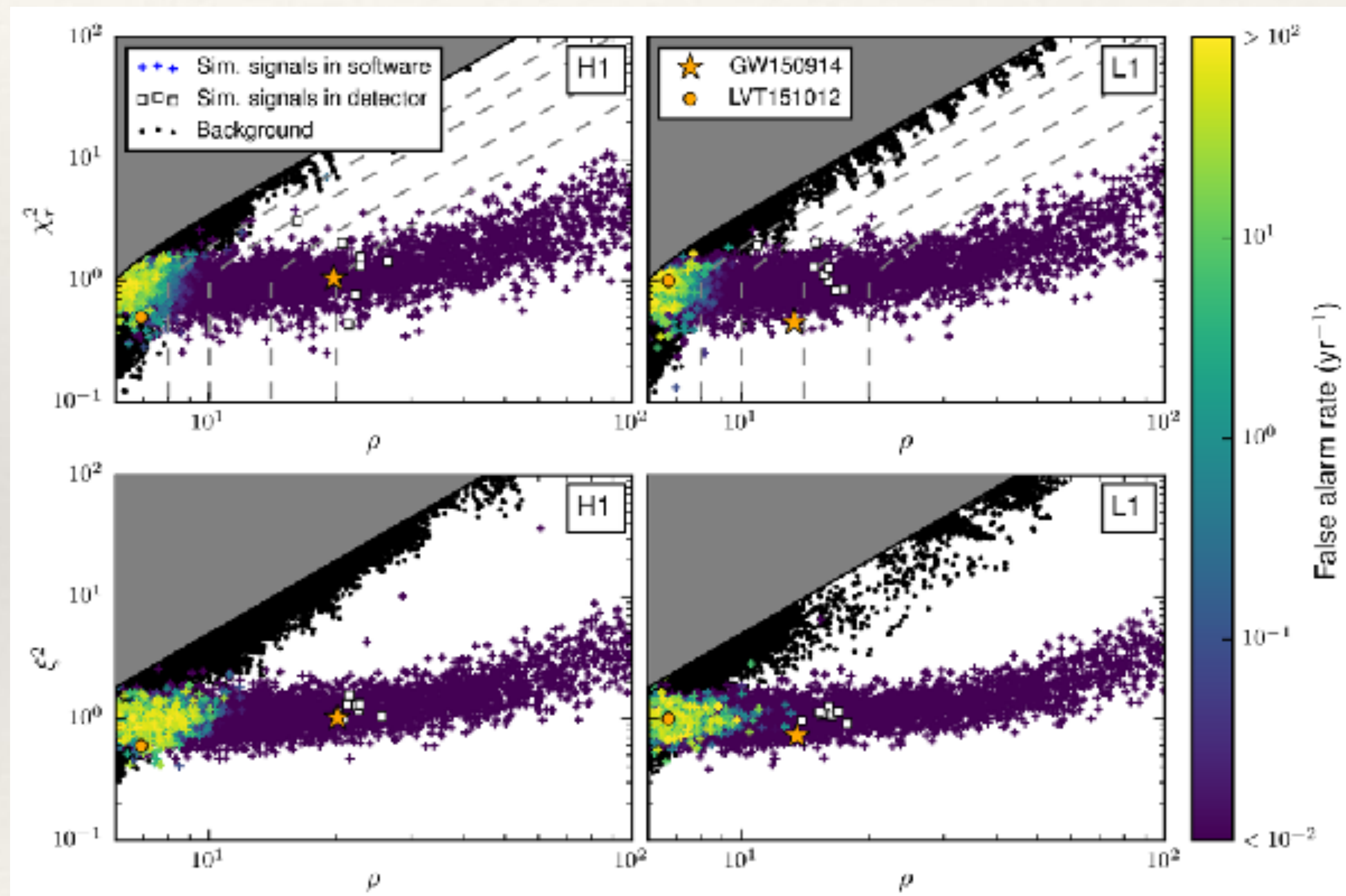
- Hence the quantity

$$\chi^2 = \sum_{k=1}^{N} \frac{|\hat{s}_k - \hat{h}_k|^2}{S_n(f_k)}$$

- follows a chi-squared distribution.

- Construct an *effective SNR* that penalises lack of fit

$$\hat{\rho} = \frac{\rho}{(1 + (\chi^2/N)^3)^{\frac{1}{6}}}$$



LVC, *Phys. Rev.* D **93**, 122003 (2016)

Search refinements: phase and Time Parameters

- Certain parameters can be maximised over cheaply, e.g., unknown phase

$$h(t; A, f_0, t_c, \phi_0) = A\cos(2\pi f_0(t - t_c) + \phi_0)$$

$$\max_{\phi_0}(s|h)^2 = A^2\left((s|h(t; A, f_0, t_c, 0))^2 + (s|h(t; A, f_0, t_c, -\pi/2))^2\right)$$

- and unknown coalescence time

$$\tilde{h}(f; A, f_0, t_c, \phi_0) = \tilde{h}(f; A, f_0, 0, \phi_0)\exp(-2\pi i f t_c)$$

$$(s|h(t; A, f_0, t_c, \phi_0)) = 2\mathrm{Re}\int_{-\infty}^{\infty}\frac{\tilde{s}^*(f)\tilde{h}(f; A, f_0, 0, \phi_0)}{S_n(f)}\exp\left(-2\pi i f t_c\right)\,\mathrm{d}f$$

- This is the inverse Fourier transform of $\tilde{s}^*(f)\tilde{h}(f; A, f_0, 0, \phi_0)/S_n(f)$. Obtain overlap for all time offsets cheaply using a Fast Fourier Transform. **Maximisation** is a frequentist approach. The Bayesian equivalent would be **marginalisation.**

The F-statistic

- The **F-statistic** extends the idea of phase maximisation to more of the extrinsic parameters. We write the waveform as a sum over four basis functions

$$h(t) = \sum_{i=1}^{4} a_i\left(\iota, \psi, D_L, \varphi_c\right) A^i\left(t; M_c, \mu, t_c, \theta, \phi\right)$$

- where

$$a_1 = \Lambda\left[\left(1+\cos^2\iota\right)\cos 2\psi\cos\varphi_c - 2\cos\iota\sin 2\psi\sin\varphi_c\right]$$
$$a_2 = -\Lambda\left[\left(1+\cos^2\iota\right)\sin 2\psi\cos\varphi_c + 2\cos\iota\cos 2\psi\sin\varphi_c\right]$$
$$a_3 = \Lambda\left[\left(1+\cos^2\iota\right)\cos 2\psi\sin\varphi_c + 2\cos\iota\sin 2\psi\cos\varphi_c\right]$$
$$a_4 = -\Lambda\left[\left(1+\cos^2\iota\right)\sin 2\psi\sin\varphi_c - 2\cos\iota\cos 2\psi\cos\varphi_c\right]$$

- and

$$A^1 = m_o\eta\, x(t)\, D^{+}\cos(\varphi)$$
$$A^2 = m_o\eta\, x(t)\, D^{\times}\cos(\varphi)$$
$$A^3 = m_o\eta\, x(t)\, D^{+}\sin(\varphi)$$
$$A^4 = m_o\eta\, x(t)\, D^{\times}\sin(\varphi)$$

The F-statistic

- Treating the amplitudes of the components of the waveform as independent, we can find the best-fit values via

$$a_i = M_{ij} N^j \qquad N^i = \langle s | A^i \rangle \qquad M_{ij} = (M^{ij})^{-1} = \langle A^i | A^j \rangle^{-1}$$

- Substituting this into the likelihood

$$\ln \mathcal{L}(\vec{x}) = \langle s \, | h(\vec{x}) \rangle - \frac{1}{2} \langle h(\vec{x}) \, | h(\vec{x}) \rangle$$

- gives

$$\mathcal{F} = \frac{1}{2} M_{ij} N^i N^j$$

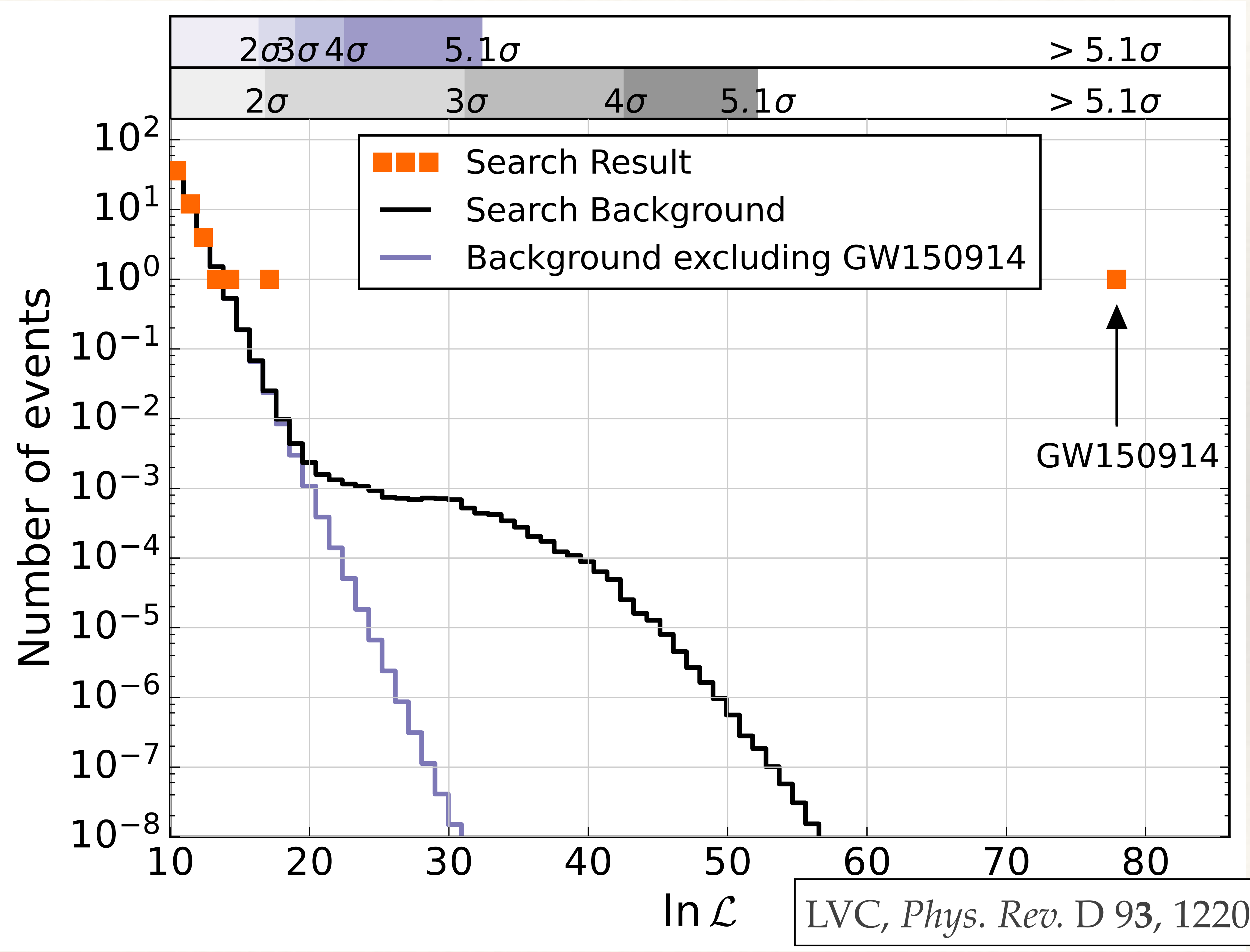
- This is the F-statistic. It depends only on intrinsic parameters, but can be used directly in a template bank instead of including templates in the extrinsic parameters as well.

- An estimate for the extrinsic parameters can be obtained from the maximized values of the a_i coefficients.

LIGO Pipelines

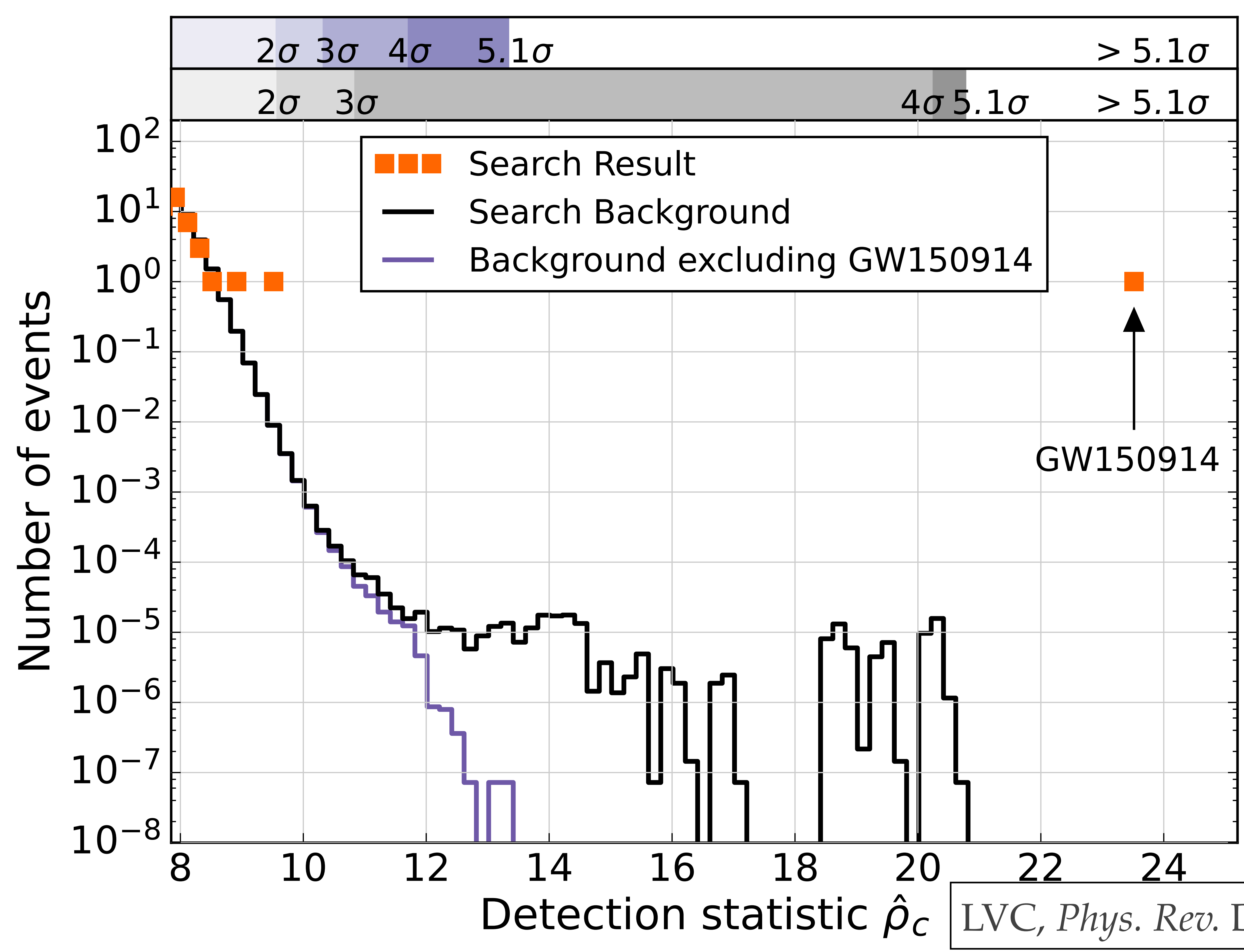
- Two main matched filtering pipelines used in LIGO for compact binary coalescence searches.
  - *pycbc*: constructs template bank of waveforms; computes chi-squared test for fit; uses effective SNR as a ranking statistic; background computed using time slides.
  - *gstLAL*: constructs template bank of waveforms, then does SVD decomposition to form a signal basis; detection statistic is likelihood ratio for signal versus noise; waveform consistency assessed by comparing SNR time series to theory; time slides again used to assess background.

LIGO Pipelines



LVC, *Phys. Rev.* D **93**, 122003 (2016)

LIGO Pipelines



LVC, *Phys. Rev.* D **93**, 122003 (2016)

PSD Estimation

- Matched filter is noise-weighted. OK if you know the noise PSD, but in general we will not. For LIGO, estimate this using off-source data.

$s_{-3} = n_{-3}$	$s_{-2} = n_{-2}$	$s_{-1} = n_{-1}$	$s_0 = h + n_0$	$s_1 = n_1$	$s_2 = n_2$	$s_3 = n_3$

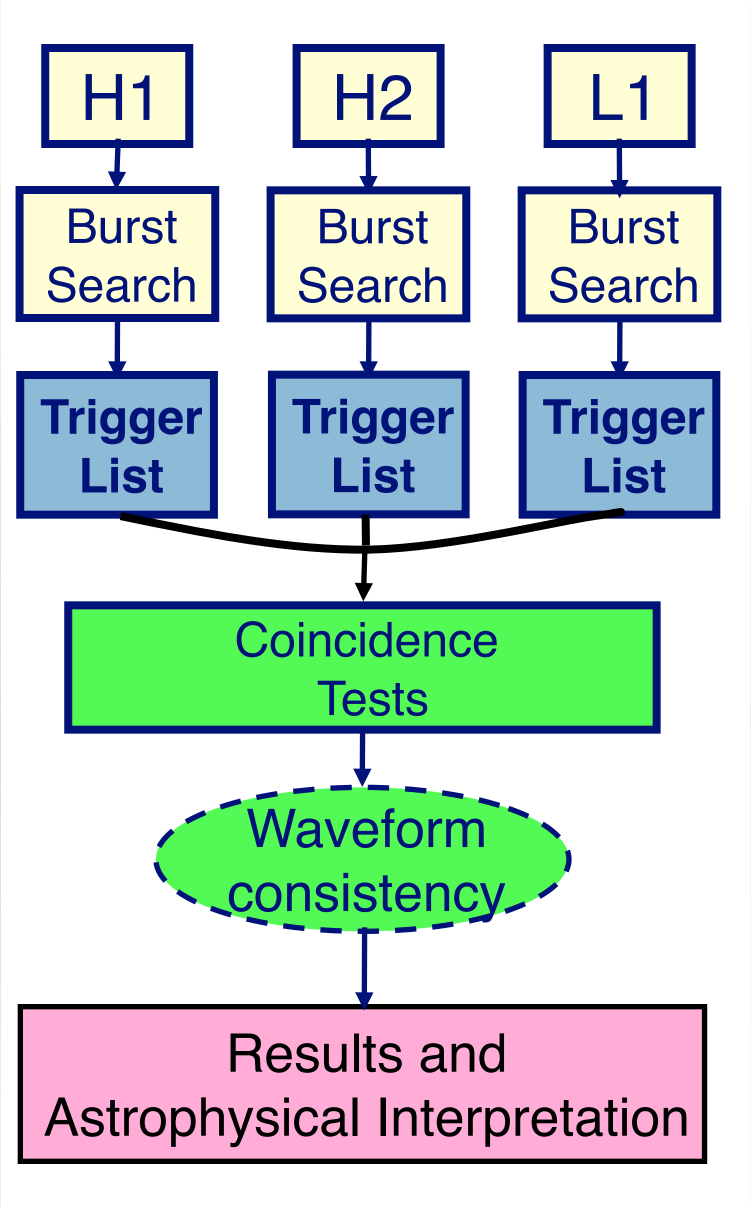
$$\sigma_0^2 \approx \frac{1}{2N} \sum_{k=1}^{N} \left(s_k^2 + s_{-k}^2 \right)$$

- In practice, use median of noise estimates, rather than the average. This is less sensitive to non-stationarities.
- No off-source data for LISA. Make progress by fitting noise and signal properties simultaneously - need reasonable noise model.

Unmodelled/Excess power searches

Unmodelled Searches

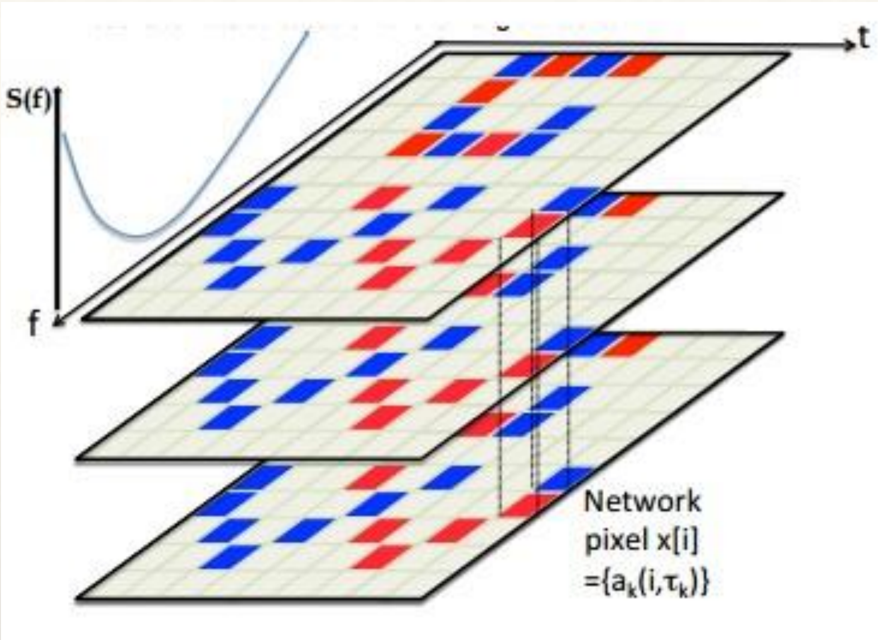
- Detection of gravitational wave bursts relies on two techniques
  - *Coincidence analysis*. As for stochastic background, combine data from multiple detectors. Likelihood of an instrumental artefact in two detectors simultaneously is small.
  - *Time-frequency analysis*. Look for changes in spectral properties over time, e.g., excess power in a set of connected pixels.
- Basic idea: construct *time-frequency* spectrograms of the data, i.e., estimate power at each frequency and time. Use spectrograms at multiple resolutions to give sensitivity to different burst morphologies.
- Look for clusters of pixels coincident between instruments.



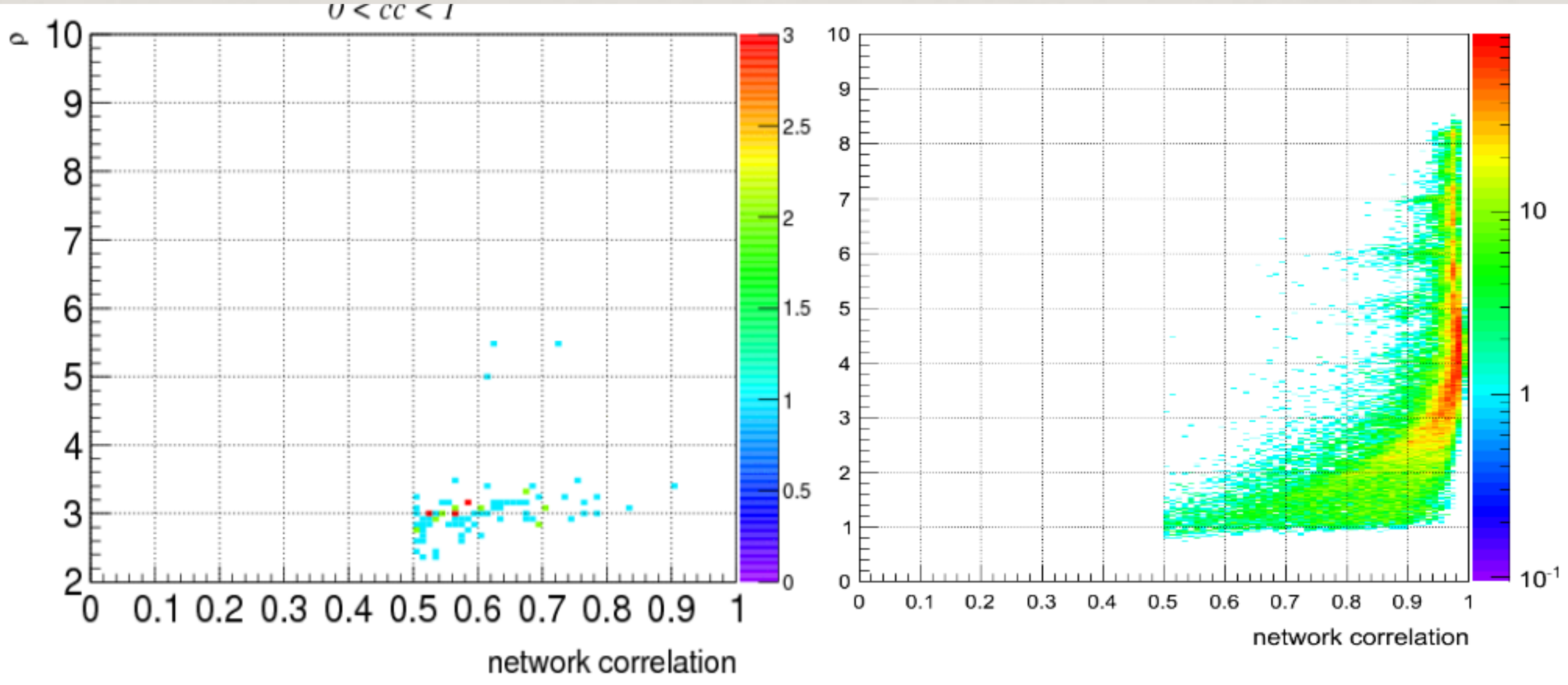
Unmodelled Searches: Coherent Wave Burst

- Combines spectrograms at multiple resolutions. Identifies pixel clusters.

- Uses various derived quantities to distinguish signals from noise artefacts, e.g., coherent and residual noise energies.

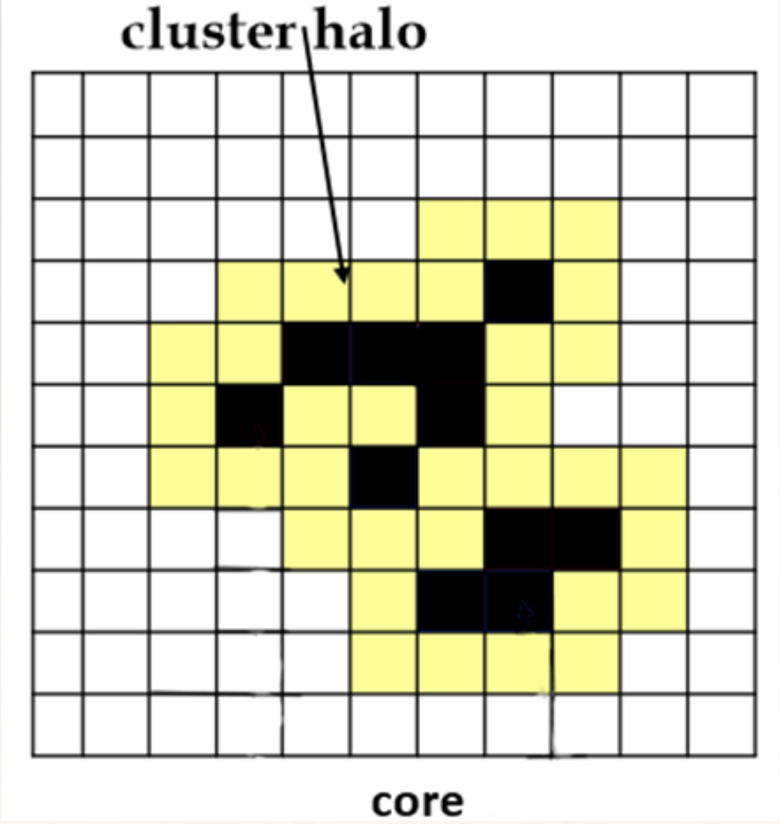


$$E_c = \sum_{m \neq n} L_{mn} \qquad N = |X - \xi_\sigma|^2 \qquad cc = \frac{E_c}{N + E_c}$$

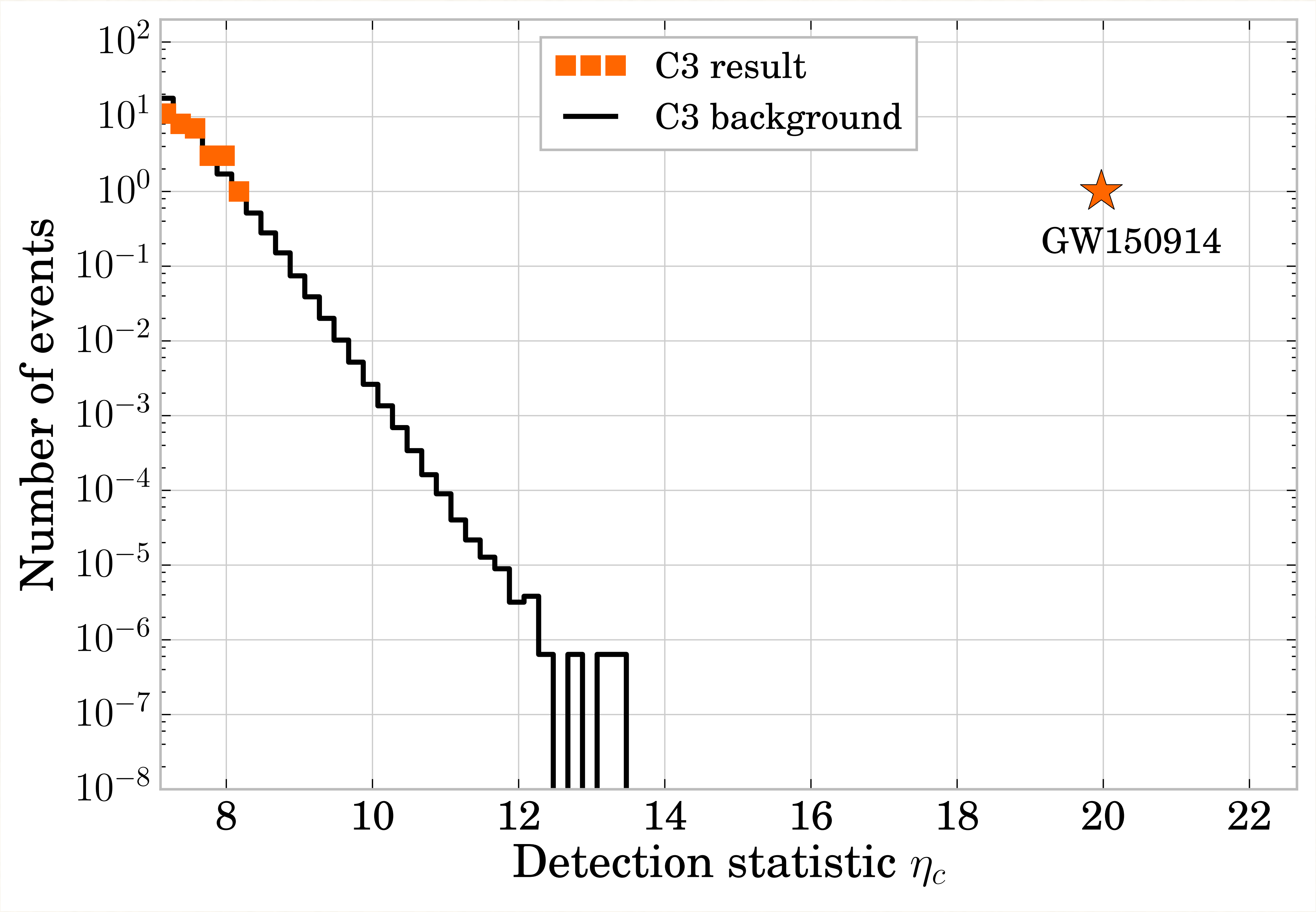


Background

Simulation

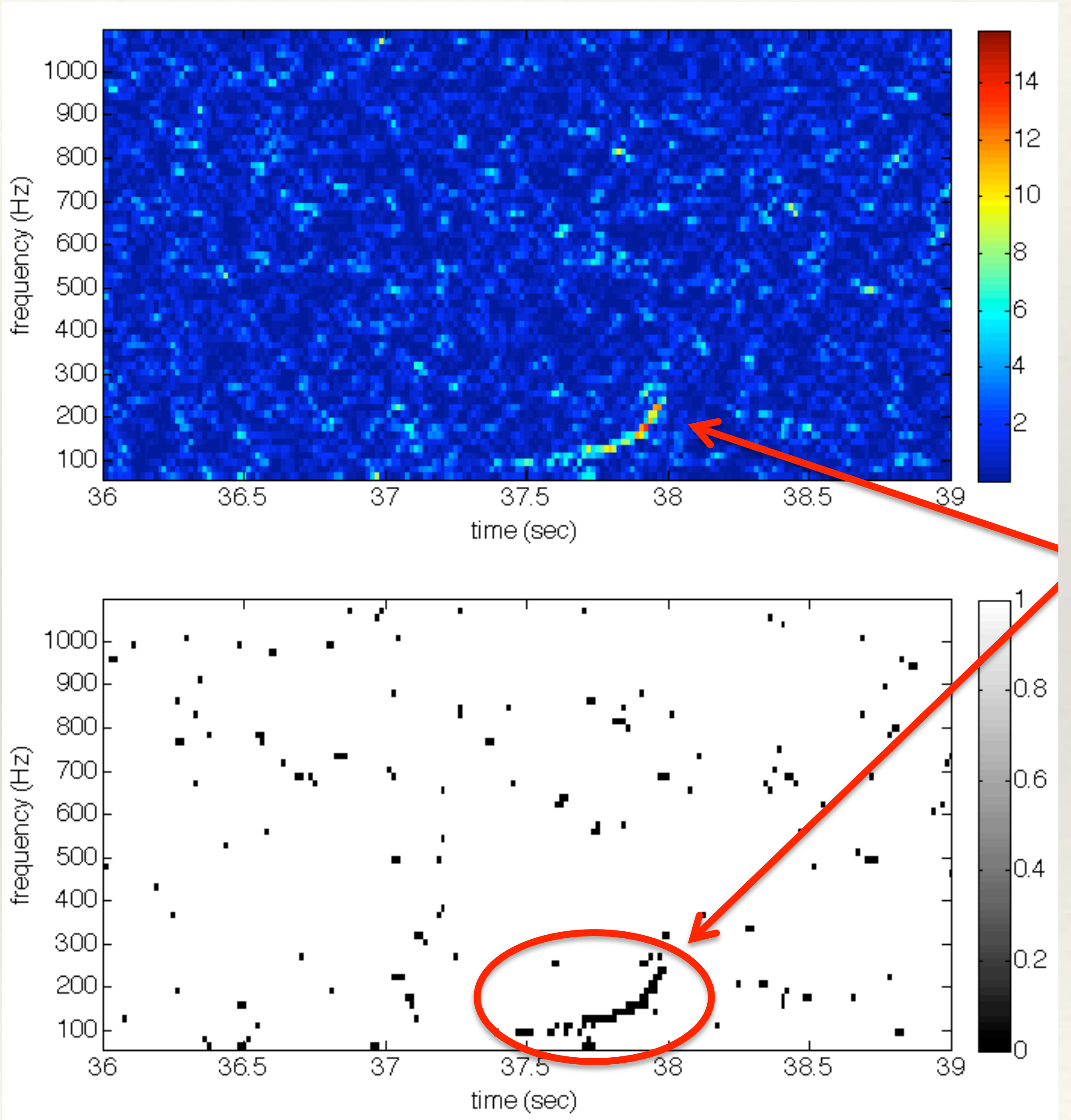


Unmodelled Searches: Coherent Wave Burst



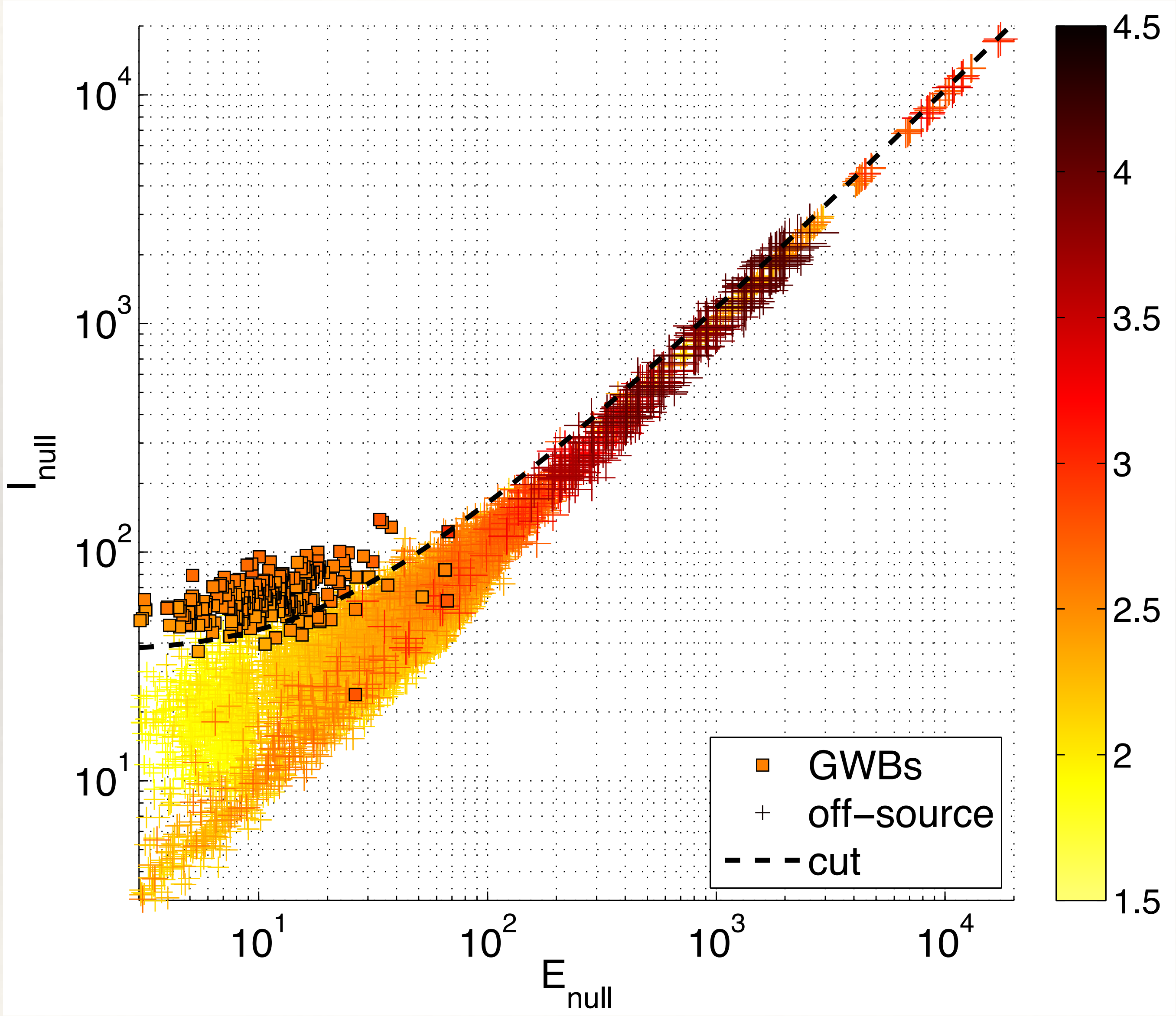
Unmodelled Searches: X-pipeline

- *X-pipeline* uses similar methods to CWB, but different implementation.
- Data is whitened and FFT'd at multiple resolutions. Data from different detectors is then summed to construct various energy measures.
- High energy (black) pixels are identified and clustered.
- Each event characterised by certain parameters - start time, peak time, duration, start frequency, peak frequency, bandwidth, number of pixels, energy measures, sky position, FFT length etc.
- Loop over sky positions and length of FFTs.



Unmodelled Searches: X-pipeline

- Analysis is in two stages. Trigger generation, as described above, then post processing.
- Post processing involves rejecting background events based on event properties, and assessment of search efficiency.
- Rejection uses different combinations of energy measures, based on randomly selected training set of injections and time slides.



Unmodelled Searches: LISA

- Time-frequency methods were also applied for EMRI detection for LISA.
    - Search for tracks in time-frequency spectrogram of data.
    - Three algorithms tried - Excess Power, HACR, CATS. Estimate detection threshold at ~2Gpc. Good parameter recovery in MLDC, but likely to fail when presented with weak or confused sources.

Unmodelled Searches: LISA

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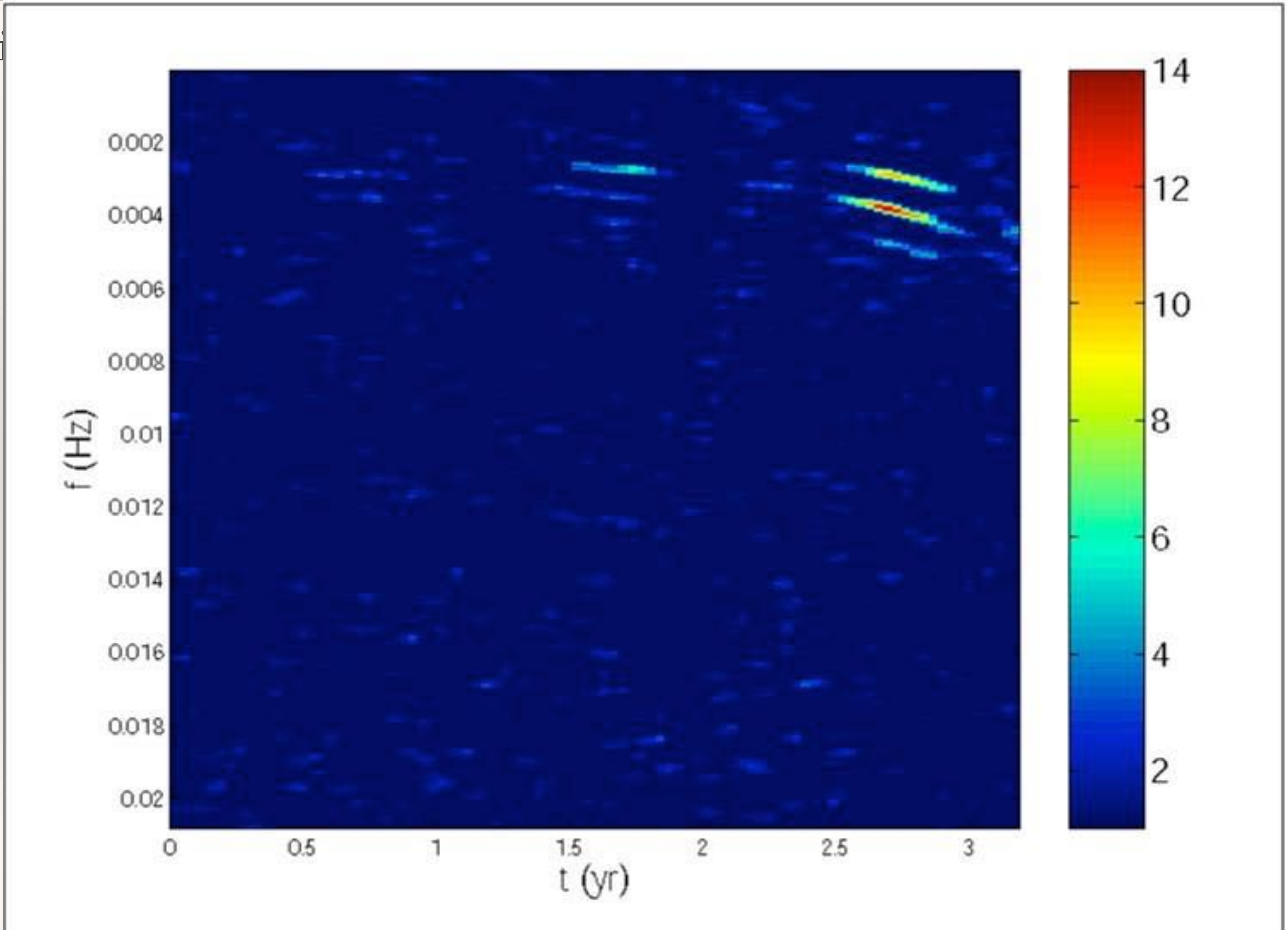

Unmodelled Searches: LISA

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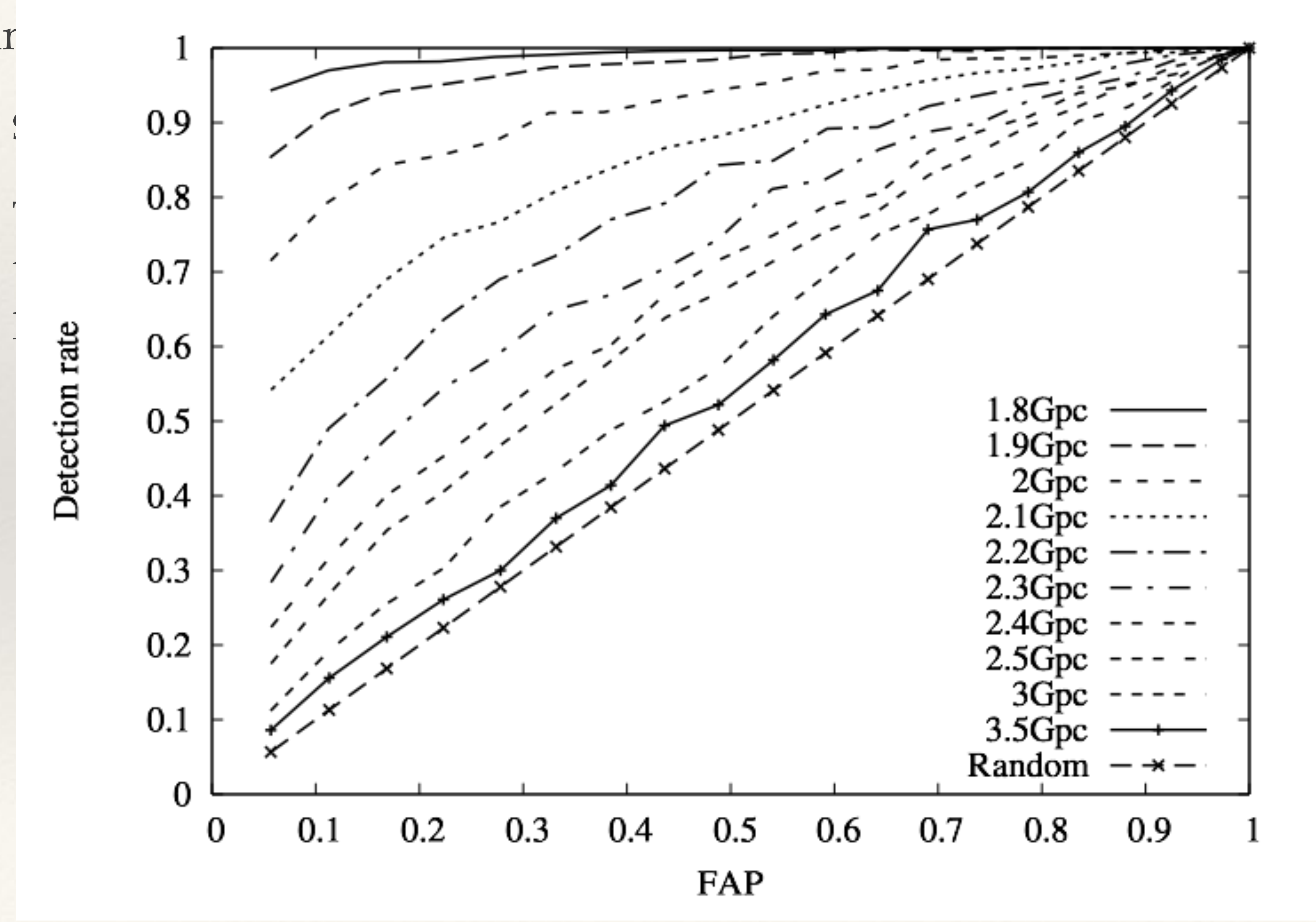


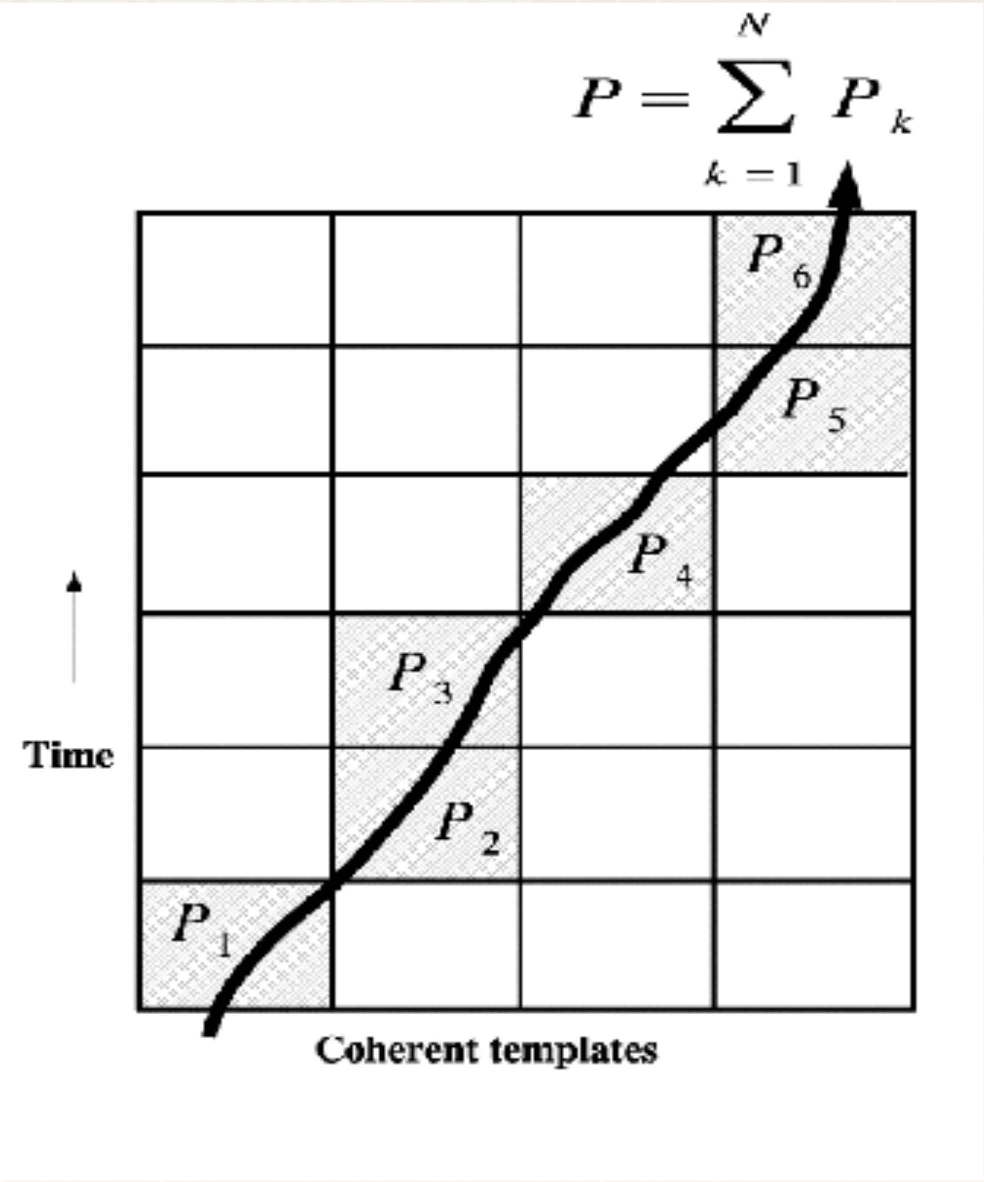
Semi-coherent search methods

Semi-coherent searches: EMRIs

- First stage is coherent matched filtering of shorter (~2 week) waveform segments. Segment length set by computational limits.
- Second stage involves incoherent summation of maximized power along trajectories through the segments.

$$\rho^2 = \sum_{\alpha=I}^{II} \sum_{i=1}^{5} \langle h_i(\lambda_I), s_\alpha \rangle^2$$

where $$\langle a, b \rangle = 4\,\Re \left[\int_0^\infty \frac{\tilde{a}^*(f)\,\tilde{b}(f)}{S_b(f)}\, \mathrm{d}f \right]$$



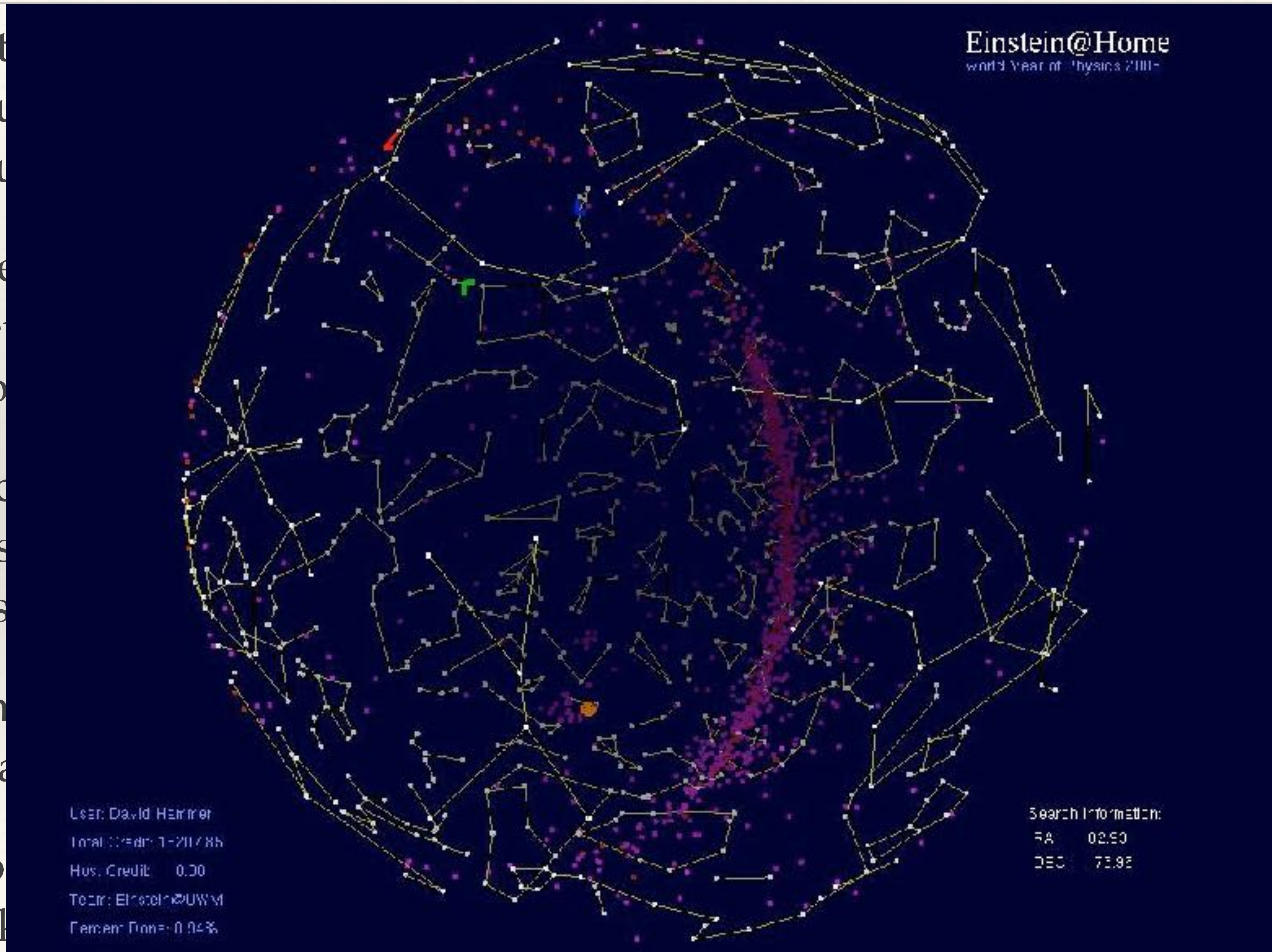
Semi-coherent searches: EMRIs

- First stage is coherent matched filtering of shorter (~2 week) waveform segments. Segment length set by computational limits.
- Second stage involves incoherent summation of maximized power along trajectories through the segments.
- Performance analysed theoretically to derive estimated EMRI event rates. Computational cost has prevented practical implementation.

Semi-coherent searches: pulsars

- LIGO unknown pulsar search also uses semi-coherent techniques.
- *Stack-Slide* algorithm is very similar to EMRI algorithm described above.
- *Hough Transform* applies the Hough Transform, a well-established technique for detecting simple shapes (edges) in an image, to the output of the coherent stage of the search.
- Requires a huge amount of computer power - Einstein@home.

- In the spirit of *Seti@home*, *Einstein@Home* is an attempt to use idle cpu hours to analyse LIGO data and assist with the unknown pulsar search. You can sign up at http://einstein.phys.uwm.edu/ !
- The program is built on BOINC (Berkeley Open Infrastructure for Network Computing) and was released in 2005 to coincide with the World Year of Physics.
- Each computer analyses a different segment of data for a particular sky position. Each data segment is farmed out to at least two nodes to ensure accuracy.
- Einstein@Home currently has approximately 500,000 active users and a total of 5GFLOPs computing power.
- No gravitational waves discovered from pulsars, but has identified unknown pulsars in other data sets.



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Searching for Backgrounds

Stochastic Gravitational Wave Fore/Backgrounds

- Stochastic backgrounds are potentially present in all frequency bands, and could therefore be seen by any of our gravitational wave detectors.
- The Polarisation of the Cosmic Microwave Background is a direct probe of cosmological gravitational waves.
- In interferometers, search for an isotropic background using cross-correlation between multiple detectors to identify common noise.

$$
\begin{aligned}
Y_Q &= \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, h_1(t_1) Q(t_1 - t_2) h_2(t_2) \\
&= \int_{-\infty}^{\infty} \mathrm{d}f \int_{-\infty}^{\infty} \mathrm{d}f' \delta_T(f - f') \tilde{h}_1^*(f) Q(f') \tilde{h}_2(f')
\end{aligned}
$$

Stochastic Gravitational Wave Fore/Backgrounds

- In the preceding equation, $\delta_T(f)$ denotes a finite time approximation to the Dirac delta function

$$\delta_T(f) \equiv \int_{-T/2}^{T/2} \mathrm{e}^{-2\pi i f t} \mathrm{d}t = \sin(\pi f T)/\pi f$$

- and $Q(t)$ denotes the cross-correlation filter. If the noise in the detectors is uncorrelated, the expectation value of depends only on the cross-correlated stochastic signal

$$\langle Y_Q \rangle = \mu = \frac{T}{2} \int_{-\infty}^{\infty} \gamma(|f|) S_{\mathrm{gw}}(|f|) \tilde{Q}(f)\, \mathrm{d}f$$

- The function $\gamma(|f|)$ is the *overlap reduction function*, which measures the loss of sensitivity due to the separation and relative orientation of the two detectors. The SNR is maximized by using the optimal filter

$$\tilde{Q}(f) \propto \frac{\gamma(|f|) S_{\mathrm{gw}}(|f|)}{S_1(|f|) S_2(|f|)} \propto \frac{\gamma(|f|) \Omega \mathrm{gw}(|f|)}{|f|^3 S_1(|f|) S_2(|f|)}$$

Stochastic Gravitational Wave Fore/Backgrounds

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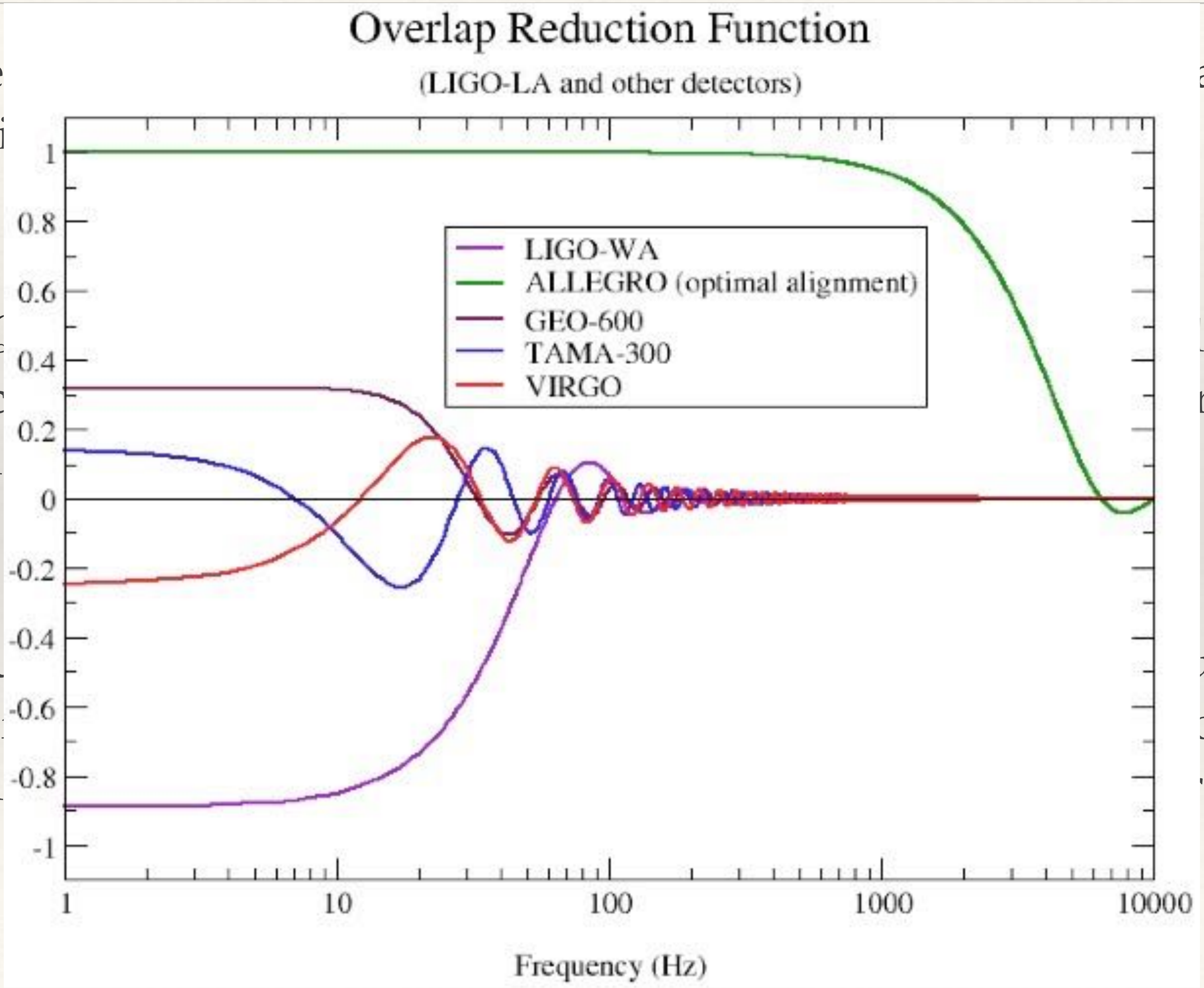
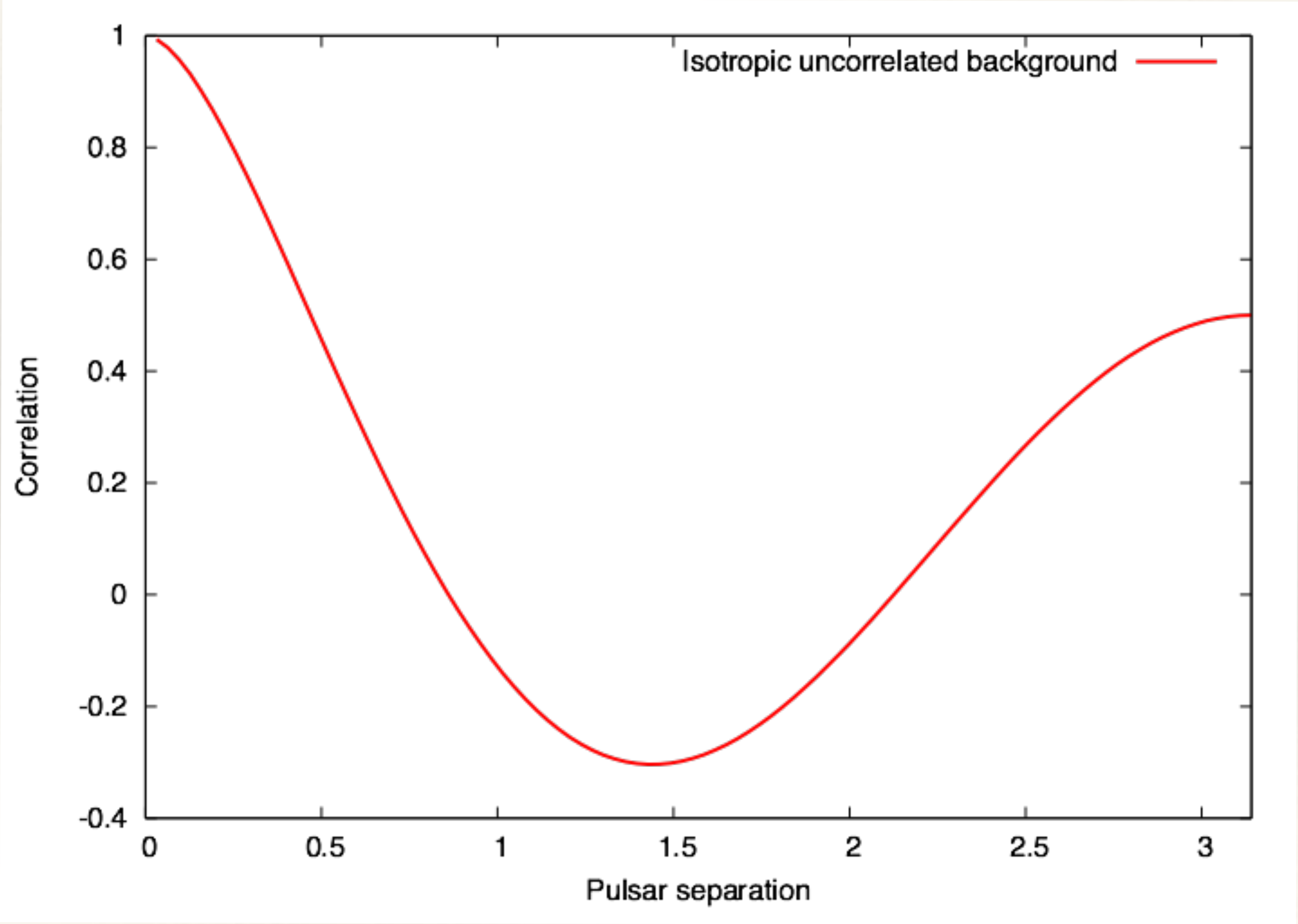


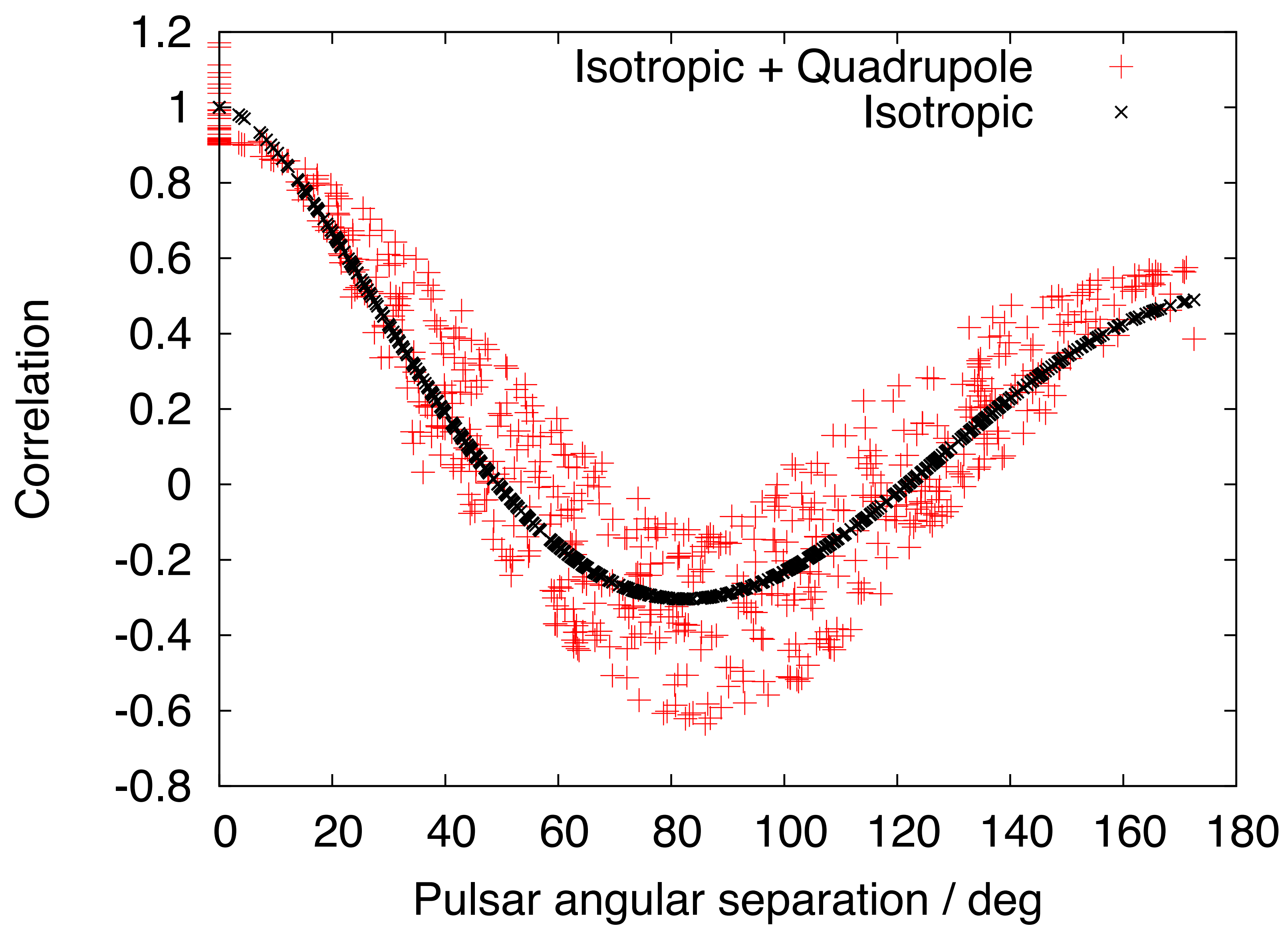
Stochastic Gravitational Wave Fore/Backgrounds

- For pulsar timing, the overlap reduction function for an isotropic background is the Hellings and Downs curve.



Stochastic Gravitational Wave Fore/Backgrounds

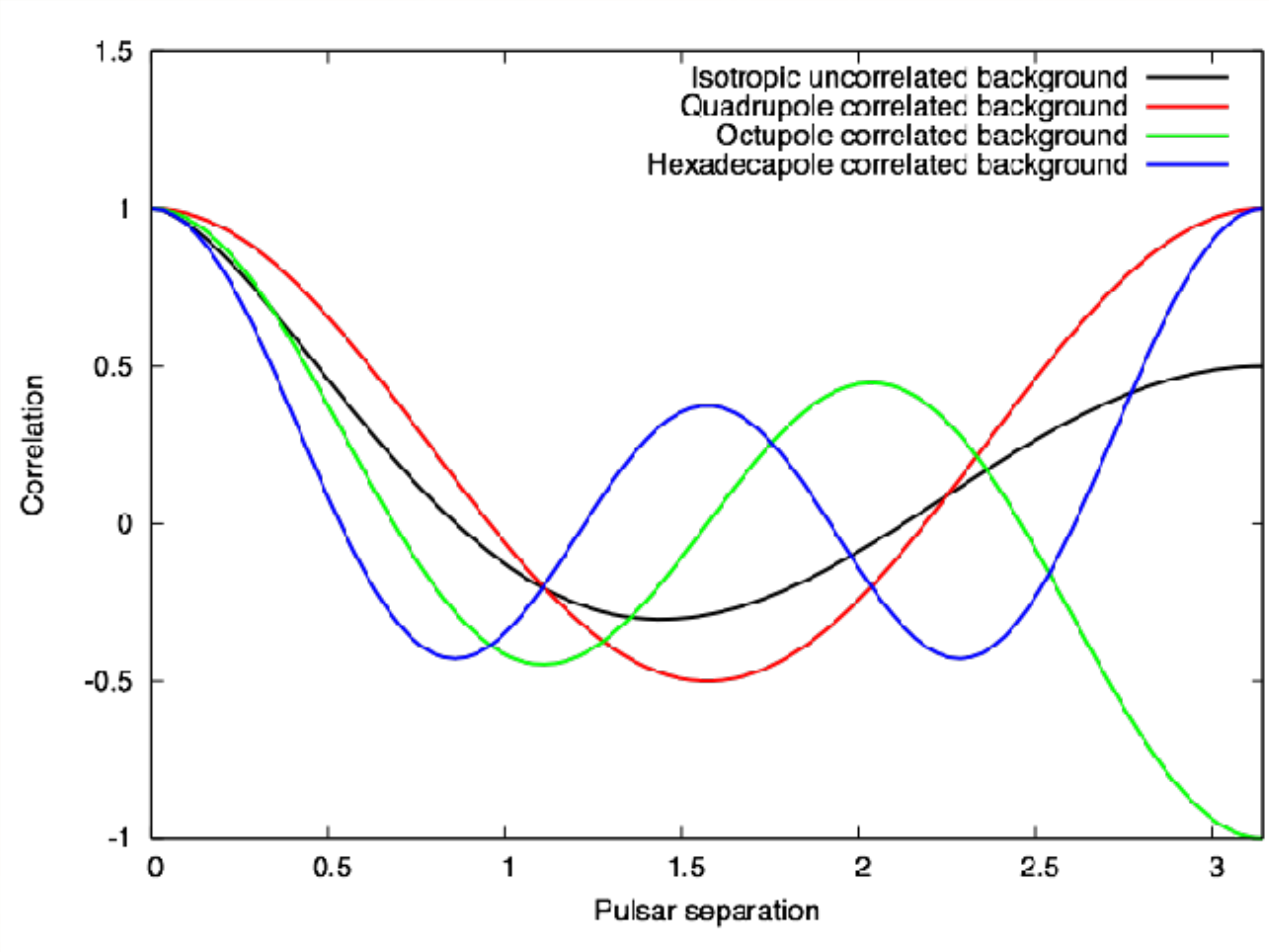
- Uncorrelated anisotropic and correlated backgrounds have different correlation functions.



Correlation function for uncorrelated quadrupole background

Stochastic Gravitational Wave Fore/Backgrounds

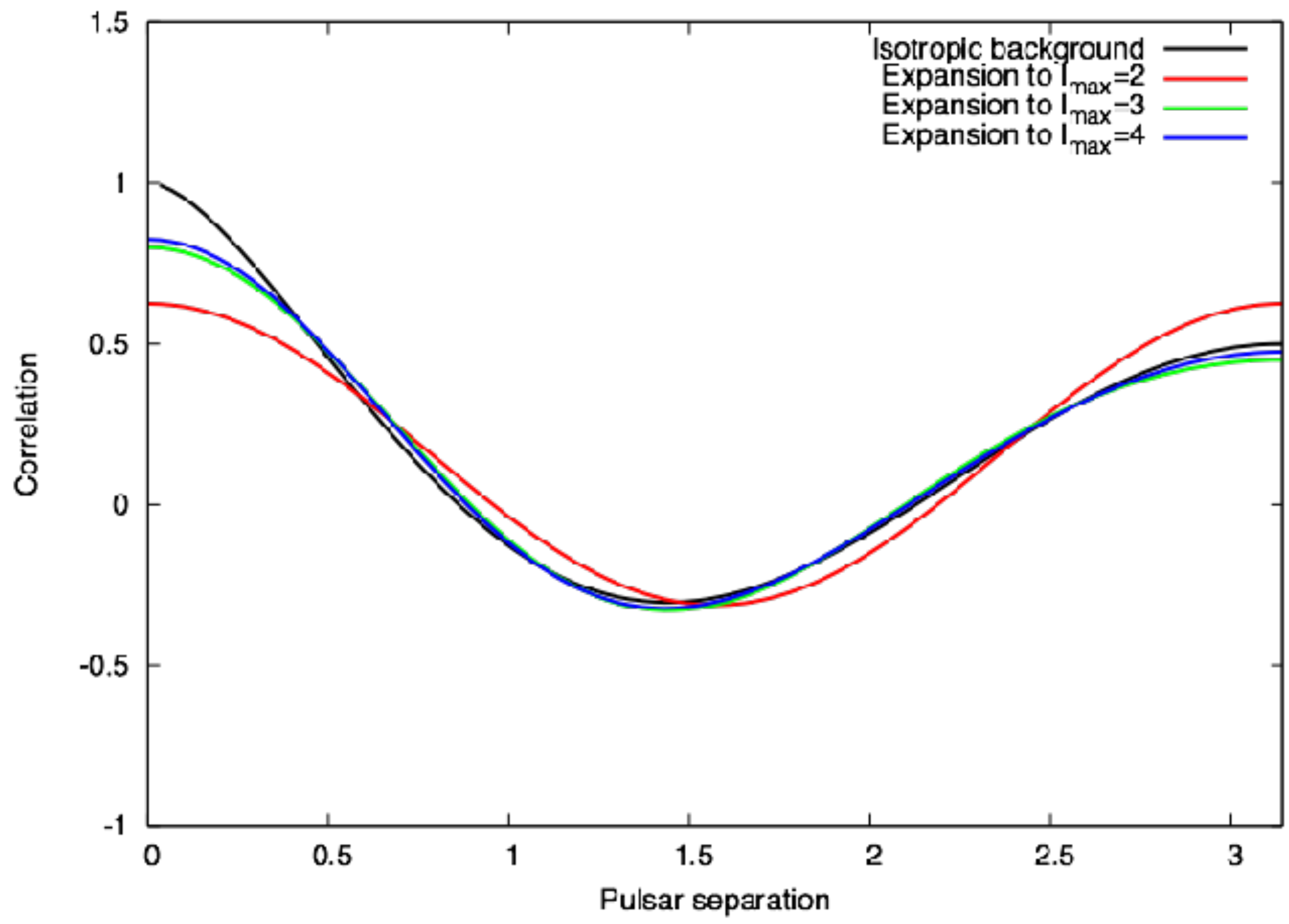
- Uncorrelated anisotropic and correlated backgrounds have different correlation functions.



Correlation function for correlated backgrounds

Stochastic Gravitational Wave Fore/Backgrounds

- Uncorrelated anisotropic and correlated backgrounds have different correlation functions.



Decomposition of isotropic uncorrelated background as a superposition of correlated backgrounds