Problem set 4

1. Suppose that a circular binary abruptly loses more than half its mass but both members of the binary retain the same velocities (in magnitude and direction) that they had prior to the mass loss. This can happen, for example, if one of the binary members undergoes a supernova which is isotropic in the original rest frame of the star. By comparing the expression for the circular speed to the expression for the escape speed, demonstrate that this abrupt loss of mass results in the binary separating and becoming two stars (the kick due to mass loss is known as the Blaauw kick, after Adriaan Blaauw). As a follow-on to that question, we note that massive stars can lose far more than half of their mass through winds that operate for hundreds of thousands to millions of years. Why wouldn't this separate all massive binaries?

2. There is moderate evidence that when a black hole forms it experiences a net kick, relative to its pre-supernova velocity, of up to ~ 100 km s^{-1}. Suppose that you have a binary which already has one black hole which is in mutual circular orbit with a star, and that the star undergoes a supernova and receives a kick of 100 km s^{-1} (here we are ignoring the Blaauw kick). Suppose that the black hole, and the pre-supernova star, had rotation axes perfectly aligned with the orbit. By how much could the orbit be tilted by a 100 km s^{-1} kick, if the original circular orbit had two 10 $M_\odot$ objects with a semimajor axis of $a = 0.1$ au, for which the gravitational wave inspiral time is just about the current age of the universe? This gives an idea for how much the orbital axis might be misaligned from the rotational axes, in this simplified scenario (although of course there are loopholes that can be exploited).

3. Now say that you start with a 30 $M_\odot$ black hole in circular orbit with an object that becomes a 30 $M_\odot$ black hole, with a semimajor axis of 1 au. That is too far, given those masses, for the binary to coalesce in the age of the universe. But suppose that upon formation of the second 30 $M_\odot$ black hole there is a 100 km s^{-1} kick (again ignoring the Blaauw kick). If that 100 km s^{-1} kick is in the right direction then the originally circular orbit acquires eccentricity, and can also reduce the inspiral time. If the kick is oriented optimally, how small a coalescence time can we get, based on the Peters equations? Without necessarily calculating, do you think that (again with optimal orientation) the minimum inspiral time after a 100 km s^{-1} kick to a circular binary is larger or smaller if the initial semimajor axis is greater than 1 au?

For the next few problems we will extend our thinking about binaries to lower-frequency binaries, particularly those involving white dwarfs, which could be seen using LISA.

4. Dr. Sane has come to you with a brilliant idea. He has realized that LISA will be the

ideal instrument to detect satellites around extrasolar planets. In particular, he envisions a $m = 6 \times 10^{26}$ g satellite (about 10% of Earth's mass, i.e., bigger than any satellite in the Solar System) orbiting with an orbital frequency of $f_{\rm orb} = 5 \times 10^{-5}$ Hz around a planet with mass $M = 2 \times 10^{31}$ g, about ten times Jupiter's mass. At gravitational wave frequencies $f_{\rm GW} < 10^{-3}$ Hz, LISA's expected spectral density sensitivity at signal to noise of 1 is $10^{-19}(10^{-3}\ {\rm Hz}/f_{\rm GW})^2\ {\rm Hz}^{-1/2}$. Assuming an observing time of 10^8 seconds, evaluate the detection prospects if the system is at a distance of 10 parsecs (about 3×10^{19} cm).

Consider a population of binaries, each of which has reduced mass μ and total mass M. Suppose they are all circular, and that the population is in steady-state, meaning that the number in a given frequency bin is simply proportional to the amount of time they spend in that bin. Also assume that the only angular momentum loss process is gravitational radiation, rather than mass transfer or other effects. For each of the following problems, derive the answers in general and then apply the numbers to WD-WD binaries, where we assume that both masses are $0.6\,M_\odot$ (note that $M_\odot = 1.989 \times 10^{33}$ g $\approx 2 \times 10^{33}$ g).

5. Using the Peters equations for circular orbits of point masses, derive the frequency $f_{\rm min}$ such that the characteristic inspiral time $T_{\rm insp} \sim 1/\left[d\ln f/dt\right]$ is equal to the Hubble time $T_H \sim 10^{10}$ yr. What is the frequency specifically for a WD-WD binary?

6. Below $f_{\rm min}$ the distribution dN/df of sources with frequency will depend on their birth population. Above it, gravitational radiation controls the distribution. Derive the dependence of dN/df on f for $f > f_{\rm min}$ (the normalization is not important).

7. Suppose there are 10^9 WD-WD binaries at frequencies $f_{\rm min} < f < 0.1$ Hz. To within a factor of 2, compute the frequency $f_{\rm res}$ above which you expect an average of less than one WD-WD binary per $df = 10^{-8}$ Hz frequency bin (this is 1/3 yr, or about the frequency resolution expected for the LISA experiment). Very roughly speaking, above $f_{\rm res}$ one can identify individual WD-WD binaries, whereas below it is the confusion limit.

8. Dr. I. M. N. Sane doesn't understand why everyone is so worried about white dwarf noise (which is supposed to be larger than the LISA instrumental noise below about 2×10^{-3} Hz). He asserts that with so many WD-WD binaries in a given bin, the total flux in gravitational waves will be very stable; in particular, he believes that from frequency bin to frequency bin, the flux will vary so little that even a weak additional source will show up easily. He comes to this conclusion by taking the square root of the flux to get a measure of the amplitude.

Show Dr. Sane the error of his ways by doing the following model problem. Let there be N sources in a given frequency bin. Suppose that they are all equally strong, but have random phases between 0 and 2π. Add the complex amplitudes based on those random phases. Take the squared magnitude of the total amplitude as a measure of the typical flux. Determine the mean and standard deviation of the flux that results. You should find that, unlike what happens when you add sources incoherently (i.e., square the amplitudes, then add), the standard deviation of the flux is comparable to the flux, so Dr. Sane's idea fails…to no one's surprise.