Problem set 3

For the first two problems, note that we are thinking about the Fermi momentum and Fermi energy to order of magnitude level only. Given that, recall that the Fermi momentum is $p_F \sim \hbar/\Delta x$, where $\Delta x \sim n^{-1/3}$ is the approximate linear space per particle for a number density n.

1. Suppose that there are μ_e degenerate particles per baryon. For example, in a white dwarf, there is roughly one proton and one neutron per electron, and electrons are the degenerate species, so $\mu_e \approx 1/2$ in that case. In a neutron star, most of the particles are neutrons and neutrons are the relevant degenerate species, so $\mu_e \approx 1$ in that case. If the degeneracy is *nonrelativistic*, compute the average energy per baryon in a star of mass M and radius R. This energy will have a positive contribution from the Fermi energy, and a negative contribution from the gravitational potential energy. Assume that the baryons, which make up most of the mass of the star, each have a mass $m_{\rm bary}$ (which is roughly the mass of a neutron or proton), and that the degenerate particles, which provide the pressure support, each have a mass $m_{\rm degen}$ (for a white dwarf, $m_{\rm degen} = m_e$, and for a neutron star $m_{\rm degen} = m_n \approx m_{\rm bary}$).

Given this, minimize the energy per particle as a function of the radius R for a fixed mass M. How does R depend on M, and how does that compare with the dependence of R on M for an object of fixed density?

2. Given the same general setup, suppose that the degeneracy is *relativistic*. If you try to minimize the energy over the radius R with fixed M, what do you find? You should find that there is a maximum stable mass; how does that mass depend on μ_e? Given your result, how would you expect the maximum masses of neutron stars and white dwarfs to compare with each other? The actual maximum gravitational mass (i.e., the mass you would derive from Kepler's laws and the orbit of a distant satellite around the star) is about 1.35 $M_\odot$ for white dwarfs and somewhere between 2 $M_\odot$ and 3 $M_\odot$ for neutron stars. Comment on this in light of your results.

3. This problem will give you a sense for the process of magnetically channeled accretion onto neutron stars. Suppose that a neutron star has a magnetic dipole moment μ, which means that at a distance r from the center of the star the magnetic field strength is $B(r) \approx \mu/r^3$ (there would, in reality, be a factor of 2 dependence on the latitude, but we will ignore that). To be simplistic, let us also suppose that matter comes in spherically at the free-fall speed $v_{\rm ff}(r) = \sqrt{2GM/r}$, with a total mass accretion rate of $\dot{M}$.

(a) Given continuity (each spherical shell of thickness dr, at any radius r, must have the same inward mass flow rate $\dot{M}$), derive the mass density $\rho(r)$ as a function of radius.

(b) The ram pressure stress is then $T_{\rm ram} \approx \frac{1}{2}\rho(r)v_{\rm ff}(r)^2$. Compare this with the magnetic stress $T_{\rm mag} \approx B(r)^2/8\pi$. At what radius are they equal? Give a numerical answer, in centimeters (note that we are using cgs units, so that B^2 has units of erg cm^{-3} if B is in Gauss), if $B = 10^8$ G (near the low end for neutron stars) and $\dot{M} = 10^{18}$ g s^{-1} (near the high end for accreting neutron stars). Compare the radius you get with $\sim 10^6$ cm, which is the order of magnitude of the radius of a neutron star.

(c) At the balance radius that you computed, what is the Keplerian rotation frequency of a circular orbit, $\nu(r) = \frac{1}{2\pi}(GM/r^3)^{1/2}$? What would typical values be for the previous case ($B = 10^8$ G and $\dot{M} = 10^{18}$ g s^{-1}, which is typical of many low-mass X-ray binaries), and for $B = 10^{12}$ G and $\dot{M} = 10^{18}$ g s^{-1}, which is typical of many high-mass X-ray binaries?

(d) We have obviously taken major liberties in our treatment! For example, the flow in most circumstances will be through a geometrically thin disk rather than through a spherical flow. Given the radial dependences that you found for $T_{\rm ram}$ and $T_{\rm mag}$, can you nonetheless make an argument that your answer for the balance radius is not likely to be tremendously far off (meaning that it is good to a factor of 10 or better)?

4. In the crust of a neutron star, the Fermi energy of electrons could be 10 MeV or more (1 MeV $\approx 1.6 \times 10^{-6}$ erg). The temperature of the crust could be 10^8 K for an old star. For those of you with some exposure to condensed matter physics, why is it that a comparison of the thermal energy (recall that $E_{\rm th} = kT$,where $k = 1.38 \times 10^{-16}$ erg K^{-1}) with the Fermi energy suggests that the crust of a neutron star should have extremely high thermal and electrical conductivity? This, combined with the superfluidity of the core, means that other than the top $\sim 10 - 100$ meters of the star, a neutron star is nearly isothermal.

5. Dr. Sane has once again demonstrated his universal brilliance, this time in his analysis of the structure of neutron stars. He realized that establishment scientists have foolishly forgotten to include the effects of superfluidity and superconductivity in their calculations of neutron star masses and radii. His calculations show that when those effects are included, the radius of a neutron star at a given mass, and the maximum mass of a neutron star, are both changed by $\approx 20\%$. The Nobel Committee has asked you to look into this discovery.

To evaluate Dr. Sane's latest claim, note that the pairing gap for superfluids and superconductors in neutron stars is likely to be ~ 0.1 MeV; this means that, energetically, a superconducting state has ~ 0.1 MeV per nucleon lower energy than a non-superconducting state. The number density of nucleons in neutron star cores is of the order of 10^{39} cm^{-3}; using this, please compute the approximate neutron Fermi energy. Given that, what do you think of Dr. Sane's estimate of the net influence of superfluidity and superconductivity on neutron star masses and radii?

Challenge: Suppose that you are doing radio observations of a double pulsar system, in which both neutron stars are visible as pulsars. We'd like to determine, qualitatively, how overdetermined the system is. That is, we'd like to know how many aspects of the system can be measured, versus how many parameters there are. This is a deliberately vague question to get you thinking about the process of measurement. If more quantities can be measured than there are parameters, the system is overdetermined and the underlying theory can be tested.