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Course webpage: `https://imprs-gw-lectures.aei.mpg.de/2022-gravitational-waves/`

Homework due date: Homeworks must be emailed by Monday, December 12 2022 to the corresponding Tutor for this homework.
Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

RECOMMENDED READINGS:

1. B. F. Schutz and C. M. Will, Astrophys.J.Lett. **291** (1985), L33-L36
2. V. Ferrari and B.Mashhoon, Phys. Rev. **D30** (1984), 295
3. E. E. Flanagan and T. Hinderer, Phys. Rev. D **77**, 021502 (2008) [arXiv:0709.1915].

I. NEWTONIAN QUADRUPOLAR TIDAL IMPRINT IN THE GW PHASING

Consider a neutron star-black hole binary system of total mass M and reduced mass μ whose orbital motion is described by Newtonian gravity. The Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} - \frac{1}{2}Q_{ij}\mathcal{E}_{ij} + L_{\rm int}, \tag{1}$$

where $L_{\rm int}$ describes the internal dynamics of the quadrupole and the Newtonian tidal field is

$$\mathcal{E}_{ij} = -m_{\rm BH}\partial_i\partial_j(1/r) = -m_{\rm BH}(3n^in^j - \delta^{ij})/r^3, \tag{2}$$

where $n^i = x^i/r$ is a unit vector. Note that $n^in_i = 1$ and $\delta^{ij}\delta_{ij} = 3$. Assume that the quadrupole is adiabatically induced and given by

$$Q^{\rm ad}_{ij} = -\lambda\mathcal{E}_{ij}, \tag{3}$$

where λ is the tidal deformability parameter. The internal Lagrangian then describes only the elastic potential energy $L^{\rm ad}_{\rm int} = -Q_{ij}Q^{ij}/(4\lambda)$. Throughout this exercise, assume that tidal effects are small and can be treated as linear perturbations.

(a) Obtain the equations of motion for r and ϕ from the Euler-Lagrange equations.

(b) Assume that the orbit is circular ($\ddot{r} = 0$ and $\dot{\phi} = \Omega$). Starting from the radial equation of motion, express the radius as $r(\Omega) = M^{1/3}\Omega^{-2/3}(1 + \delta r)$ and compute the linear tidal corrections δr.

(c) Calculate the energy of the system from (1). Specialize to adiabatic quadrupoles and circular orbits, and express the energy in terms of Ω.

(d) The leading order gravitational radiation is generated by the total quadrupole of the system $Q_{ij}^{T} = Q_{ij}^{\rm orbit} + Q_{ij}$. Compute the tidal contribution to the energy flux from the quadrupole formula.

(e) In the stationary phase approximation (SPA) for the gravitational wave signal, the phasing can be computed from the formula

$$\frac{d^2\Psi_{\rm SPA}}{d\Omega^2} = 2\frac{dE/d\Omega}{\dot{E}_{\rm GW}}. \tag{4}$$

Compute the tidal contribution to $\Psi_{\rm SPA}$, to linear order in the tidal effects. Express your result in terms of the post-Newtonian parameter $x = (M\Omega)^{2/3} = (\pi M f_{\rm GW})^{2/3}$ and show that the tidal phase correction scales as x^5 relative to the leading order phasing.

II. BLACK-HOLE QUASI-NORMAL MODES

In the lectures and the previous tutorial session, it was shown that the quasinormal modes (QNMs) of a Schwarzschild black hole are characterized by complex frequencies $\omega = \omega_R + i\omega_I$, with ω_R and ω_I the real and the imaginary parts, respectively.

(a) Use Table I from arXiv:gr-qc/0411025 to plot ω_R and ω_I of the quadrupolar mode ($l = 2$) versus n, where n is the overtone number that identifies the number of nodes in the radial wavefunction (plus 1 in the reference's conventions). Use $n = 1$–12, 20, 30, 40. [Note that the values in Table I correspond to $(\omega_R, -\omega_I)$ in our conventions, given the time-dependence of the QNMs as $e^{i\omega t}$.]

Your plot should exhibit some features which could be considered strange according to certain intuition, interpreting ω_R as an oscillation frequency and ω_I as a decay rate. For typical systems with a set of vibrational modes, like a string or an elastic body, both the oscillation frequency and the decay rate increase with increasing overtone number, i.e. with an increasing number of nodes in the wavefunction. The QNM plot, however, shows that ω_R is first decreasing with n, then has a zero, and then increases to an asymptotically constant value. This behavior can be seen as somewhat less mysterious by reinterpreting ω_R and ω_I as follows.

(b) Consider a simple damped oscillator with amplitude $\psi(t)$ obeying

$$\ddot{\psi} + \gamma_0\dot{\psi} + \omega_0^2\psi = 0. \tag{5}$$

Writing the two linearly independent solutions as $\exp((\pm i\omega_R - \omega_I)t)$, find the relationship between ω_R, ω_I and ω_0, γ_0. Invert this relation, make plots of ω_0 and γ_0 versus n for the Schwarzschild QNMs and comment how this interpretation alleviates the above discussion.