Gravitational-Wave Course Homework Sheet 4

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Course webpage: `https://imprs-gw-lectures.aei.mpg.de/2022-gravitational-waves/`

Homework due date: Homeworks must be emailed by Thursday, November 24 2022 to the corresponding Tutor for this homework.
Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

RECOMMENDED READINGS:

1. Neutron star physics: `http://adsabs.harvard.edu/abs/2004Sci...304..536L`
2. A. Buonanno and T. Damour, Phys. Rev. **D59** (1999) 084006.
3. A. Buonanno and T. Damour, Phys.Rev. **D62** (2000) 064015.

EXERCISES:

I. GRAVITATIONAL WAVES FROM PULSARS

Neutron stars possess a rigid crust that is 10 billion times stronger than steel and can support a "mountain" of up to ∼few cm height. Consider a neutron star rotating with angular frequency Ω around a principal body axis $\mathbf{e}_3$ and with constant principal moments of inertia I_1, I_2, I_3. Assume that the neutron star has a deformation such that $I_1 \neq I_2$.

1. Consider the inertia tensor $I_{ij} = \int d^3x\rho\left(r^2\delta_{ij} - x^i x^j\right)$ that is given by $I_{ij} = \mathrm{diag}(I_1, I_2, I_3)$ in the body frame whose axes rotate with the neutron stars. Compute the components of the inertia tensor in an inertial frame. Use the analogy between I_{ij} and the Newtonian quadrupole moment to obtain the power radiated in gravitational waves. Express your result in terms of the ellipticity ϵ and I_3, where
$$\epsilon = \frac{I_1 - I_2}{I_3} \tag{1}$$
2. Consider a neutron star that is approximated as a uniform density sphere with mass $\sim 1.4M_\odot$ and $R \sim 10\mathrm{km}$ so that $I_3 \sim \frac{2}{5}MR^2 \sim 10^{45}\mathrm{g\,cm}^2$. Its rotational energy is $E = I_3\Omega^2/2$. For the Crab pulsar, the rotational period is $P = 33\mathrm{ms}$. Use the balance between the energy radiated in gravitational waves and the change in E to obtain its spin-down rate $\dot{\Omega}$. Show that for a fiducial ellipticity of $\epsilon = 10^{-7}$ the rate of change in the frequency is small and thus the GWs are approximately monochromatic over ∼few years observation time.

3. The observed spindown rate of the Crab pulsar is $\dot{P} = 4.2 \times 10^{-13} s/s$. Assuming that this is caused solely by GW emission, what would the ellipticity of the Crab pulsar need to be to explain this value?

   In several pulsars, the spindown rate has been measured with pulsar timing observations and is generally quantified by a braking index n defined by $\dot{\Omega} \propto \Omega^n$. For the Crab pulsar, $n \approx 2.5$ ($n \sim 3$ is expected for magnetic dipole radiation), while for the Vela pulsar $n \approx 1.5$. Read off the braking index from your result (b). Is GW emission the dominant mechanism for the spindown of the Crab pulsar?

II. ON THE EFFECTIVE-ONE-BODY HAMILTONIAN AND DYNAMICS

We have derived in class the mapping between the *real* PN Hamiltonian and the *effective* Hamiltonian using the Hamilton-Jacobi formalism. Here we want to construct the effective-one-body (EOB) Hamiltonian using a canonical transformation.

Using reduced (or dimensionless) variables $\mathbf{Q}, \mathbf{P}$ and $\hat{H}_{\rm eff}$, the EOB Hamiltonian reads

$$\hat{H}_{\rm eff}(Q,P) = c^2 \sqrt{A(Q)\,\left[1 + \frac{1}{c^2}\,\mathbf{P}^2 + \left(\frac{A(Q)}{D(Q)} - 1\right)\,\frac{1}{c^2}\,(\mathbf{N}\cdot\mathbf{P})^2\right]}\,, \tag{2}$$

where $\mathbf{N} = \mathbf{Q}/Q$ and

$$\begin{aligned} A(Q) &= 1 + \frac{a_1}{c^2\,Q} + \frac{a_2}{c^4\,Q^2} + \frac{a_3}{c^6\,Q^3} + \cdots\,, &(3)\\ D(Q) &= 1 + \frac{d_1}{c^2\,Q} + \frac{d_2}{c^4\,Q^2} + \cdots\,, &(4)\end{aligned}$$

where a_i, d_i are unknown coefficients that will be determined by the mapping to the (reduced) PN Hamiltonian

$$\begin{aligned} \hat{H}_{\rm real}(q,p) &= \hat{H}_{\rm Newt}(q,p) + \frac{1}{c^2}\,\hat{H}_{\rm 1PN}(q,p) + \cdots\,, &(5)\\ \hat{H}_{\rm Newt}(q,p) &= \frac{1}{2}\mathbf{p}^2 - \frac{1}{q}\,, &(6)\\ \hat{H}_{\rm 1PN}(q,p) &= -\frac{1}{8}(1-3\nu)\,\mathbf{p}^4 - \frac{1}{2q}\,[(3+\nu)\,\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2] + \frac{1}{2q^2}\,, &(7)\end{aligned}$$

where $\mathbf{q}$ and $\mathbf{p}$ are reduced variables, $\mathbf{n} = \mathbf{q}/q$ and $\nu = m_1\,m_2/(m_1+m_2)^2$, being m_1 and m_2 the black-hole masses. At 1PN order the real and effective Hamiltonians are related as

$$1 + \frac{\hat{H}_{\rm real}(q,p)}{c^2}\left(1 + \alpha_1\,\frac{\hat{H}_{\rm real}(q,p)}{c^2}\right) = \frac{\hat{H}_{\rm eff}(Q(q,p),P(q,p))}{c^2}\,, \tag{8}$$

where α_1 is an unknown coefficient that will be determined by the mapping. The canonical transformation at 1PN order is

$$\begin{aligned} Q^i &= q^i + \frac{1}{c^2}\frac{\partial G_{\rm 1PN}}{\partial p_i}\,, &(9)\\ P_i &= p_i - \frac{1}{c^2}\frac{\partial G_{\rm 1PN}}{\partial q^i}\,, &(10)\end{aligned}$$

with

$$G_{\rm 1PN}(\mathbf{q},\mathbf{p}) = (\mathbf{q}\cdot\mathbf{p})\,\left[c_1\mathbf{p}^2 + \frac{c_2}{q}\right]\,, \tag{11}$$

where c_1, c_2 are unknown coefficients that will be determined by the mapping.

The goal of this exercise is to determine α_1, c_1, c_2 as a function of ν.
Insert the canonical transformation given in Eqs. (9) and (10) in Eq. (8) and expand the latter in PN orders through 1PN order. By equating terms with the same structures in $\mathbf{q}$, $\mathbf{p}$, derive the equations for the unknown coefficients a_1, α_1, c_1, c_2 and set $a_2 = a_3 = ... = a_n = d_1 = d_2 = ... = d_n = 0$. In this case you should find that: $\alpha_1 = \nu/2$, $c_1 = -\nu/2$ and $c_2 = 1 + \nu/2$. [Hint: introduce the parameter $\epsilon^2 \equiv 1/c^2$, work with the square of Eq. (8) to get rid of the square root in Eq. (2), and neglect the terms with order higher than $O(\epsilon^4)$. Note that it is sufficient to derive $Q \equiv |\mathbf{Q}| = \sqrt{Q^i\, Q_i}$, $P \equiv |\mathbf{P}| = \sqrt{P^i\, P_i}$ and $\mathbf{N} \cdot \mathbf{P} = N^i\, P_i$ as function of $q \equiv |\mathbf{q}|$, $p \equiv |\mathbf{p}|$ and $\mathbf{n} \cdot \mathbf{p}$ through 1PN order using the canonical transformation given in Eqs. (9) and (10).]