# Gravitational-Wave Course — Homework Sheet 3

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**Course webpage:** https://imprs-gw-lectures.aei.mpg.de/2022-gravitational-waves/

**Homework due date:** Homeworks must be emailed by Thursday, November 17 2022 to the corresponding Tutor for this homework.
**Homework rules:** Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

**Grading system:** The homework sheet will be graded with an overall score within *0, 1, 2*.

*0*: not sufficient, the student has done less than half of the problems and did not attempt all of them.

*1*: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

*2*: good, the student correctly solved almost all the problems.

**RECOMMENDED READINGS:**

1. post-Newtonian approximation and EFT: M. Levi, Rept. Prog. Phys. **83**, 075901 (2020); arxiv:1807.01699.
2. Fokker action: T. Damour, G. Esposito-Farese, Phys. Rev. D **53** 5541–5578 (1996); arXiv:gr-qc/9506063.

**EXERCISES:**

1. **Central-force problem at 1PN order**

   Starting from the 1PN-Lagrangian (also known as the Einstein-Infeld-Hoffman Lagrangian)

$$\mathcal{L} = \mathcal{L}_{\rm N} + \mathcal{L}_{\rm 1PN} + \mathcal{O}(c^{-4}), \tag{1a}$$

$$\mathcal{L}_{\rm N} = \frac{m_1}{2}\mathbf{v}_1^2 + \frac{m_2}{2}\mathbf{v}_2^2 + \frac{Gm_1m_2}{r}, \tag{1b}$$

$$\begin{aligned}\mathcal{L}_{\rm 1PN} = &\frac{1}{8c^2}m_1\mathbf{v}_1^4 + \frac{1}{8c^2}m_2\mathbf{v}_2^4 - \frac{G^2m_1m_2(m_1+m_2)}{2c^2r^2} \\ &+ \frac{Gm_1m_2}{c^2r}\left(\frac{3}{2}\mathbf{v}_1^2 + \frac{3}{2}\mathbf{v}_2^2 - \frac{7}{2}\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_1\cdot\mathbf{n}\,\mathbf{v}_2\cdot\mathbf{n}\right),\end{aligned} \tag{1c}$$

   in the coordinates $\mathbf{r}_1 \equiv \mathbf{x}_1$, $\mathbf{r}_2 \equiv \mathbf{x}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$, where $r = |\mathbf{r}_1 - \mathbf{r}_2|$, $\mathbf{n} = (\mathbf{r}_1 - \mathbf{r}_2)/r$:

   a) Derive the canonical momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. [Recall from classical mechanics that $\mathbf{p}_a = \partial\mathcal{L}/\partial\mathbf{v}_a$.] Then, introduce the variables $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{P} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, and $\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1)/2$, and show that $\mathbf{P}$ is conserved.

   b) Obtain the relative-motion Hamiltonian $H = \mathbf{p}_1\cdot\mathbf{v}_1 + \mathbf{p}_2\cdot\mathbf{v}_2 - \mathcal{L}$ at 1PN order in the variables $\mathbf{r}$, $\mathbf{p}$, $M = m_1 + m_2$ and $\nu = m_1\,m_2/M^2$. [Hint: in carrying out the calculation here and below keep only terms at 1PN order! It is also strongly suggested to use Mathematica to manipulate long algebraic expressions.]

c) Compute the binding energy $E = H$ and orbital angular momentum $L$ at 1PN order for circular orbits. Express the final result for $E$ and $L$ in terms of the velocity $v \equiv (M\Omega)^{1/3}$, where $\Omega$ is the orbital frequency. [Hint: Impose the circular orbit condition and derive the relation between $r$ and $\Omega$. You will find a few new terms at 1PN order beyond the usual Newtonian relation $M/r^3 = \Omega^2$. You might find it convenient to work with Hamilton's equations in spherical coordinates and choose the motion to be in the equatorial plane.]

d) Compute the periastron advance at 1PN order for nearly circular orbits. [Hint: It is more convenient to employ the relative-motion Lagrangian. Use the conservation of energy and angular momentum to derive the equation for the radial perturbation around a circular orbit and compute the radial frequency $\Omega_r$ as function of $\Omega$. The fractional advance of the periastron per radial period is $\Delta\Phi/(2\pi) = K(\Omega) - 1$, where $K(\Omega) = \Omega/\Omega_r$.] **[optional!]**

e) Study the stability of circular orbits using the 1PN Hamiltonian. **[optional!]**

Consider the polar coordinates $(r, \phi, p_r, p_\phi)$ and a perturbation of the circular orbit defined by

$$\begin{aligned} p_r &= \delta p_r \,, \\ p_\phi &= p_\phi^0 + \delta p_\phi \,, \\ r &= r_0 + \delta r \,, \\ \Omega &= \Omega_0 + \delta\Omega \,, \end{aligned}$$

where $r_0$, $\Omega_0$ and $p_\phi^0$ refer to the unperturbed circular orbit. Write down the Hamilton equations and linearize them around the circular orbit solution. You should find

$$\begin{aligned} \delta\dot{p}_r &= -A_0\,\delta r - B_0\,\delta p_\phi \,, \\ \delta\dot{p}_\phi &= 0 \,, \\ \delta\dot{r} &= C_0\,\delta p_r \,, \\ \delta\Omega &= B_0\,\delta r + D_0\,\delta p_\phi \,, \end{aligned} \tag{2}$$

where $A_0, B_0, C_0$ and $D_0$ depend on the unperturbed orbit. Determine explicitly $A_0, B_0, C_0$ and $D_0$.

Look at solutions of Eqs. (2) proportional to $e^{i\sigma t}$ and find the criterion of stability. [Hint: you should find that there exists a combination $\Sigma_0$ of $A_0, B_0, C_0$ and $D_0$ such that when $\Sigma_0 > 0$ the orbits are stable. The innermost stable circular orbit (ISCO) corresponds to $\Sigma_0 = 0$].

Express $\Sigma_0$ as function of $v = (M\Omega)^{1/3}$ and show that for any value of the binary mass ratio the ISCO at 1PN order coincides with the Schwarzschild ISCO. [This is an accident, which does not hold at high PN orders!]

Finally, show that $\Sigma_0 = 0$ coincides with $\Omega_r = 0$. What is the physical meaning of this result?

2. **Leading-order spin-orbit interaction**

We consider the action of two *spinning* point-particles labeled $a = 1, 2$ interacting gravitationally to order $c^{-2}$:

$$\begin{aligned} S = &\frac{1}{32\pi G}\int dt d^3x \left(4\phi\Delta\phi - \frac{1}{c^2}A_i\Delta A_i\right) \\ &+ \sum_{a=1,2}\int dt\left[-m_a\phi + \frac{1}{c^2}\left(\frac{3}{2}v_a^i S_a^{ij}\partial_j\phi + m_a v_a^i A_i + \frac{1}{2}S_a^{ij}\partial_i A_j\right)\right]_{\mathbf{x}=\mathbf{x}_a} \\ &- \frac{1}{8\pi G c^2}\int dt d^3x\, \phi\partial_t^2\phi + \sum_{a=1,2}\int dt\, m_a\left[-c^2 + \frac{\mathbf{v}_a^2}{2} + \frac{1}{c^2}\left(\frac{\mathbf{v}_a^4}{8} - \frac{3\mathbf{v}_a^2}{2}\phi - \frac{1}{2}\phi^2\right)\right]_{\mathbf{x}=\mathbf{x}_a} + \mathcal{O}(c^{-4}). \end{aligned} \tag{3}$$

Here the spin tensor is given by $S_a^{ij}(t) = -S_a^{ji}(t) = \epsilon_{ijk}S_a^k(t)$ and the spin vector fulfills the usual angular-momentum Poisson bracket $\{S_a^i, S_a^j\} = \epsilon_{ijk}S_a^k$ (a corresponding kinematic term in the action encoding this Poisson bracket is omitted for simplicity). As in the lectures, let us eliminate the gravitational fields $\phi$ and $A_i$ from the action, keeping contributions up to order $c^{-2}$, but now pick out the leading-order spin-orbit interactions, that is, terms that are linear in $\mathbf{S}_1$ and independent of $\mathbf{S}_2$:

- The last line in (3) does not contribute to the desired result. Briefly argue why, or annotate in the following steps where those terms would enter and show that they either contribute to higher orders in $c^{-2}$ or to the nonspinning case.
- Derive the field equations for $\phi$ and $A_i$ from the action.
- Eliminate the fields by inserting the solution to the field equations into the action, keeping only terms to the desired order in spins and $c^{-2}$. As in the lectures, you can make use of the formula $\Delta^{-1}\delta(\mathbf{x} - \mathbf{x}_a) = -1/(4\pi|\mathbf{x} - \mathbf{x}_a|)$ to get an explicit result for the two-body Lagrangian, and drop singular self-interactions. (The result is a so-called Fokker action.)
- **Optional:** From this Lagrangian, obtain the relative-motion Hamiltonian in the center-of-mass system where $\mathbf{p}_1 = -\mathbf{p}_2$, with definitions given in the previous exercise (the spin variables remain untouched in the Legendre transform).

The analogous result linear in $\mathbf{S}_2$ and independent of $\mathbf{S}_1$ can be obtained by swapping particle labels. **Optional:** Include in the above calculation the leading-order $S_1$-$S_2$ interaction, that is, terms containing both $\mathbf{S}_1$ and $\mathbf{S}_2$.

Note that the spin of black holes can be written as $S_a = Gm_a^2\chi_a/c$ where the dimensionless spin $\chi_a$ is less than 1. Hence each order in spin is suppressed by another power of $c^{-1}$. The leading-order spin-orbit interaction for compact binaries is then at order $c^{-3}$ or 1.5PN, and the $S_1$-$S_2$ interaction at order $c^{-4}$ or 2PN.