

**Instructor:** Alessandra Buonanno (alessandra.buonanno@hu-berlin.de)

**Tutor** (corresponding for this sheet): Lorenzo Pompili (lorenzo.pompili@aei.mpg.de)

**Tutor:** Raj Patil (raj.patil@aei.mpg.de)

**Course webpage:** <https://imprs-gw-lectures.aei.mpg.de/2022-gravitational-waves/>

**Homework due date:** Homeworks must be emailed by Monday 10/31/2022 to the corresponding Tutor for this homework.

**Homework rules:** Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

**Grading system:** The homework sheet will be graded with an overall score within 0, 1, 2.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

#### Recommended readings:

1. LIGO Scientific Collaboration and Virgo/ Collaboration: Observation of Gravitational Waves from a Binary Black Hole Merger. Phys. Rev. Lett. 116(6), 061102 (2016). <https://arxiv.org/pdf/1602.03837.pdf>.
2. LIGO Scientific Collaboration and Virgo Collaboration: The basic physics of the binary black hole merger GW150914. Annalen der Physik, 529. <https://arxiv.org/pdf/1608.01940.pdf>.
3. A. Einstein: Über Gravitationswellen. In: Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin (1918), 154–167. (English translation available at <http://einsteinpapers.press.princeton.edu/vol7-trans/25>.)

#### I. BASIC PHYSICS OF GW150914:

The goal of this exercise is to understand the basic formulas of gravitational-wave (GW) physics and how they can be used to extract information from the first GW observation. Numbers of equations and figures below refer to arXiv:1608.01940.

##### a) Power emitted in GWs:

Starting from Newtonian physics and Einstein's quadrupole formulas (4) and (5), fill in the details of the derivation of Eq. (A4), the rate at which energy is emitted in the form of GWs from a binary on a circular orbit.

In particular, give the intermediate expression for the GW strain tensor  $h_{ij}$  as a function of time.

b) *Frequency evolution and estimating the mass scale:*

Combine the previous result with more Newtonian physics to derive Eq. (A5), the relationship between the orbital frequency, its rate of change, and the masses. Derive also Eq. (8).

This makes a certain assumption to allow combining Newtonian physics (which predicts a constant energy) with the quadrupole formula (which predicts that the energy decreases). Explain the assumption and check its applicability in this context.

Use the result and Fig. 2 (which is essentially a Fourier transform of Fig. 1) to estimate the chirp mass, and the total mass assuming a 1:1 mass ratio, for GW150914.

c) *Drag force instead of GWs?:*

What if the bodies in a binary experienced drag forces due to a gaseous environment? Could this explain the observed frequency-time dependence (prior to the peak), instead of energy loss due to GW emission? Assuming the drag force acting on the orbiting bodies is proportional to  $v^p$  where  $v$  is their velocity and  $p$  is some power (which is typically 1 or 2), derive an analog of Eq. (A5), up to proportionality, and go on to find the frequency as a function of time, i.e. find  $\alpha$  and  $\beta$  for the relationships  $\dot{\omega} \propto \omega^\alpha$  and  $\omega^\beta \propto t$ . Compare these with the values for GW emission.

Extract the data points from Fig. 3, and obtain an estimate for  $\beta$  from the slope of a linear fit on a log-log plot.

Does a drag force provide a viable alternative explanation of the frequency-time behavior?

d) *Orbital separation, why BHs?:*

Given the estimate of the (chirp) mass scale from above, use the GW frequency and Newtonian physics to estimate the distance between the centers of the orbiting bodies, first at the beginning of the signal, and then at the location of the peak amplitude. Explain why the latter estimate strongly suggests that the objects are black holes.

e) *Distance to the source:*

Use the magnitude of the GW strain from Fig. 1 and your formula for the GW strain  $h$  from Problem Ia derive an order-of-magnitude estimate for the distance of the source from the Earth.

How does this compare to the luminosity distance derived in Sec. 5?

## II. ORDERS OF MAGNITUDE OF GRAVITATIONAL-WAVE STRENGTH FOR SOURCES ON THE EARTH:

Using the leading the quadrupolar term  $G \ddot{I}_2 / (c^4 r)$ , estimate the amplitude of GWs produced from the following earth-based events:

- A meteorite having diameter of 2 km and density of  $\rho = 4000 \text{ kg/m}^3$  and hitting the ground at a speed of 25 km/sec;
- A big chunk of piezoelectric material (mass  $M = 10 \text{ kg}$ ) driven to oscillation at a frequency of 100 MHz with amplitude  $A = 10^{-3} \text{ m}$ .

To derive the above results take into account that GWs exist only in the so-called wave zone, i.e., at a distance from the source which is at least equal to the reduced wavelength  $\lambda = c / (2\pi f)$ .

**III. MULTIPOLAR EXPANSION OF THE FAR-ZONE GRAVITATIONAL FIELD:**

In the lecture the gravitational field is written in a multipolar expansion. Show that it is dimensionless. Assuming that the gravitational field is proportional to  $G/r$  (where  $r$  is the distance between the observer and the source, and  $G$  is the Newton constant), and that each term of the expansion depends only on  $c$  and on derivatives of the multipole mass-moments  $I_L$  and current-moments  $J_L$ , employ dimensional analysis to show that the multipolar expansion is unique.