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Problem Set 3, NR

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1 Initial data for a spinning black hole

Let's solve the initial data problem in one of the simplest, yet numerically non-trivial settings. We consider puncture initial data for a single black hole with spin[1]
Let us first collect all equations needed for the numerical solution: As laid out in class, we assume conformal flatness, i.e. in Cartesian coordinates

$$\tilde{\gamma}_{ij} = \delta_{ij}, \tag{1}$$

vanishing of the trace of the extrinsic curvature ($K = 0$), and we take the conformal trace-free extrinsic curvature from the Bowen-York solution with spin. That is, in Cartesian coordinates,

$$\tilde{A}^{ij} = \frac{6}{r^3} n_{(i}\varepsilon_{j)kl} S^k n^l, \tag{2}$$

where $r = |x^i| = \sqrt{\delta_{ij}x^i x^j}$, $n^i = x^i/r$. In Cartesian coordinates with conformal flatness, upper and lower indices are equivalent, so index position in Eq. (2) is irrelevant. Also, the ε_{ijk}-symbol takes the values ± 1.
In the puncture data approach, one writes the conformal factor as

$$\psi = \frac{m}{2r} + 1 + u, \tag{3}$$

where u is a finite and continuous function on $\mathbf{R}^3$ which we will be solving for. The parameter m=const is the 'bare mass' of the puncture; it can be set to be $m = 1$ throughout this problem. In terms of u, the Hamiltonian constraint simplifies to

$$\Delta_f u = -\frac{1}{8}\frac{r^7 \tilde{A}^{ij}\tilde{A}_{ij}}{(r + m/2 + ur)^7}, \qquad \vec{x} \in \mathbf{R}^3, \tag{4}$$

with the asymptotic condition

$$u \to 0, \text{ as } r \to \infty. \tag{5}$$

Furthermore, the square of $\tilde{A}^{ij}$ simplifies to

$$\tilde{A}^{ij}\tilde{A}_{ij} = \frac{18}{r^6}\left(S^2 - (S^i n^i)^2\right) = \frac{18 S^2 \sin^2\theta}{r^6}, \tag{6}$$

[1] This problem draws on results summarized in [2]. And of course, puncture data was first proposed by Brandt & Brügmann [1] while they were at AEI.

where in the second equality we assumed that the spin is parallel to the z-axis, $S^i = (0, 0, S)$, and where θ is the usual polar angle, $\cos\theta = z/r$. Substituting this into Eq. (4), we finally arrive at

$$\Delta_f u = -\frac{9S^2}{4}\frac{r\sin^2\theta}{(r + m/2 + ur)^7}, \qquad \vec{x} \in \mathbf{R}^3. \tag{7}$$

Our goal now is to solve Eq. (7) subject to boundary condition (5). We will do so with finite differences.

In Cartesian coordinates, the Laplacian is simply $\Delta_f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, which is well-suited to discretization by finite-differences. To get going, let us therefore disregard that our problem is axisymmetric and instead finite-difference directly the 3-D Cartesian problem. Take a 3-dimensional equally spaced grid with N^3 grid-points covering a cube $[-a, a]^3$ centered on the origin. That is

$$\vec{x}_{ijk} = (-a + ih, -a + jh, -a + kh), \tag{8}$$

with $h = 2a/(N - 1)$. The solution is represented by the vector $\underline{u}$ consisting of the values of u at the grid-points, $u_{ijk} = u(x_{ijk})$.
We will need to derive a matrix-equation

$$\mathbf{A}\underline{u} = \underline{\rho}, \tag{9}$$

where $\mathbf{A}$ encodes the derivative operators and the Dirichlet-boundary conditions, and $\underline{\rho}$ represents the right-hand-sides. While filling $\underline{u}$ and $\mathbf{A}$, you will have to choose an ordering of the 3-D grid-points inside the one-dimensional vector $\underline{u}$, for instance row-major, where the 3-D index (i,j,k) is represented by the 1-D index $K = k + Nj + N^2 i$. Let us derive and test this matrix-representation first:

Task FD-1: Discretization
Use 2nd order finite-difference stencils to express $\Delta_f u$ as a matrix-operation $\mathbf{A}\underline{u}$. For the points on the boundary, assume Dirichlet conditions, i.e. set the corresponding row of $\mathbf{A}$ to be all zeroes with 1 on the diagonal.
Code in your favourite programming language. Test & debug your code by applying it to quadratic functions of increasing complexity, for which the 2nd order FD formulae are exact (i.e. start with $u = 1$, $u = x$, $u = y$, etc)..
For the concrete function $u(x, y, z) = 2x + 3y^2 + 4xz - z^2 - 2$, plot u and $\mathbf{A}\underline{u}$ in the xz-plane.
Hints: For this task, neither the scale a nor the resolution N does matter. You can choose $a = 10$, and use a small value for N like 5 or 9.

The task FD-1 should have given you reasonable confidence that your matrix $\mathbf{A}$ is correctly implemented. Now we need to get a feeling for how much accuracy we might expect:

FD-2: Toy Laplace-Problem, part I
Choose your own problem $\Delta_f u = \rho$ where you know the solution u_0 and the right-hand-side ρ. Plot your choice of u_0 and ρ in the xz-plane.
Initialize $\underline{u}$ and $\underline{\rho}$ with the solution $u_0(x)$ and $\rho(x)$. Evaluate the residual

$$\underline{r} \equiv \mathbf{A}\underline{u} - \underline{\rho}. \tag{10}$$

For $N = 9$ and $N = 21$, plot the residual $\underline{r}$ in the xz-plane. Furthermore, evaluate the residual for different values of N and plot $||\underline{r}||$ vs N to demonstrate that your code constructing $\mathbf{A}$ and evaluating Eq. (10) is convergent at the expected rate.

Hint: Be modest, pick a function u_0 which has variations on scales comparable to the box-size $2a$, so that even with $N \sim 10 \cdots 20$ you already achieve errors of, say 10%.

So far, we have not yet solved anything, but merely verified that our discretizations are correct. Now let us actually solve the toy-problem:

FD-3: Toy Laplace-Problem, part II
Given your toy-problem represented by u_0 and ρ. Solve

$$\Delta_f u = \rho, \quad x \in [-a, a]^3 \tag{11}$$

with Dirichlet boundary conditions,

$$u(x) = u_0(x), \quad x \in \partial[-a, a]^3. \tag{12}$$

For $N = 9$ plot the finite-difference solution u in the xz-plane. Plot also u_0 and the error $u - u_0$.
Demonstrate numerical convergence of your scheme by solving with increasingly large N, and by plotting the error $|u - u_0||$ vs. N.
We recommend you do a direct matrix-inverse of $\mathbf{A}$. The computational cost will increase rapidly with N, therefore stop increasing N when an individual numerical solve takes more than 10 seconds (we expect this to happen for $N \sim 20$).

Now we are almost ready to have our first attempt at the puncture initial data, Eq. (7). The remaining difficulty is that Eq. (7) is *nonlinear*. We will again try to circumvent this difficulty with the simplest possible scheme, fixed-point iteration.

Task FD-4: solve puncture data
Use a box with size $a = 10$, set $m = 1$, and choose a reasonably small spin $S = 0.2$. Approximate the asymptotic boundary condition Eq. (5) by a Dirichlet condition $u = 0$ on the boundary of the box.
Using the tools you developed in FD-1 through FD-3, solve Eq. (7) iteratively. That is we strive to find ever better solutions $\underline{u}^{(k)}$, $k = 0, 1, 2, \ldots$ as follows:
1) initialize with $\underline{u}^{(0)} = 0$.
2) given an approximate solution $\underline{u}^{(k)}$, evaluate the right-hand-side of Eq. (7) at your grid-points and call it $\underline{\rho}^{(k)}$
3) Now solve $\mathbf{A}\underline{u}^{(k+1)} = \underline{\rho}^{(k)}$ for the improved $\underline{u}^{(k)}$
4) stop when the solutions don't change much anymore.

Plot the final solution $\underline{u}^{(k)}$ in the xz plane.

You have just solved for a spinning BH!
Now let us work towards the maximum spin achievable by black holes in puncture initial data.
A single spinning BH is axi-symmetric. In axi-symmetry, angular momentum cannot be radiated, and therefore, the angular momentum of the black hole remains constant and equal to the parameter S if the initial data set is evolved. In contrast, energy stored in the initial data slice will either be absorbed by the black hole, or emitted to infinity in the form of gravitational waves. As has been shown by actual evolutions, most of this excess energy falls into the black hole, which increases in mass. If we assume that *all* energy is absorbed by the black hole, then the final mass of the black hole will equal $E_{\rm ADM}$. Then, the dimensionless BH spin after the initial data has relaxed to a stationary state would be $\chi_{\rm final} \approx \epsilon_J$, where

$$\epsilon_J \equiv S/E_{\rm ADM}^2. \tag{13}$$

Usually, $E_{\rm ADM}$ is computed at infinity from the asymptotic fall-off of the metric. Since our current solutions only extend to $a = 10$, we can't really extract this asymptotic fall-off. Instead, we can resort to the accidental property of inversion symmetry of a single spinning puncture black hole: The region near the puncture, $r \to 0$, corresponds to a second asymptotically flat universe. For a single spinning puncture black hole, the two asympotically flat ends ($r \to \infty$ and $r \to 0$) are related to each other by inversion symmetry at a sphere with radius $R_{\rm inv} = m/2$. Specifically, the transformation $r \to R_{\rm inv}/r^2$ is an identity operation, so that both asymptotically flat ends have the same ADM-energy, which can be shown[2] to be

$$E_{\rm ADM} = 1 + u(0). \tag{14}$$

Task FD-5: maximum relaxed spin
Evaluate $u(0)$ from your solution to FD-4, and thus compute ε_J. You should find a value close to 0.2.
Solve for increasingly large S, and plot ε_J vs. S. Are you able to see the turn-over where ε_J grows more slowly than S, approaching a limit smaller than unity for large S? Compare to Figure 2 of [2].

[2] [2], page 5

FD-6: Open-ended continuations (optional)
The problems so far are impacted by two significant sources of error: the fairly small value of N achievable in 3D with our simple direct inversion of the matrix $\mathbf{A}$. And second, the quite small value of a, which cannot represent the asymptotic fall-off $u \to 0$ as $r \to \infty$. Here are a few suggestions if you are interested to continue improving on this problem set:

- Estimate the relative importance of N and a on the error for ε_J. Is there a different choice for the two that leads to smaller errors?
- Exploit axisymmetry and code suitable 2-D stencils and regularity conditions on the axis.
- If you don't like to worry about regularity conditions, exploit the $z \to -z$ symmetry and axisymmetry to solve only in one octant, $x \in [0, a]^3$. You will need zero-slope boundary conditions on the coordinate-axis planes, however, you are gaining a factor 8 in reduction in grid-points which you can use to increase either a or reduce h.
- Represent the solution as a series in spherical harmonics. Due to axisymmetry, only $m = 0$ contributes, eliminating many terms. Moreover, spherical harmonics are eigenfunctions under the angular piece of the Laplacian, so that different l terms in Δ_f decouple. This turns $\Delta_f u = \rho$ into a sequence of 1-D radial problems, one for each l.

References

[1] Steven Brandt and Bernd Bruegmann. A Simple construction of initial data for multiple black holes. *Phys. Rev. Lett.*, 78:3606–3609, 1997.

[2] Geoffrey Lovelace, Robert Owen, Harald P. Pfeiffer, and Tony Chu. Binary-black-hole initial data with nearly-extremal spins. *Phys. Rev. D*, 78:084017, 2008.