# Numerical Hydrodynamics

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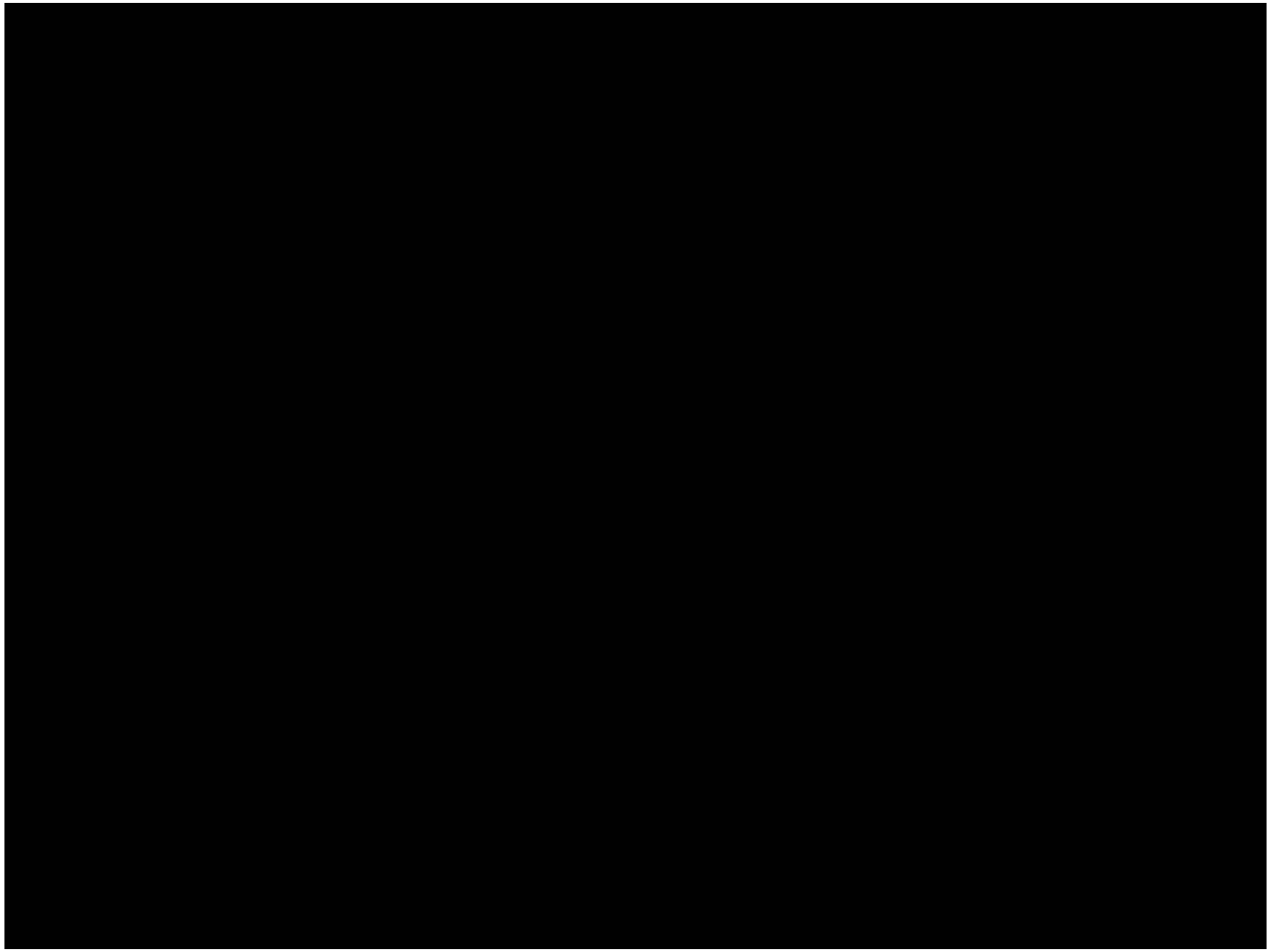
# 1. Introduction

## Why do we need a numerical hydrodynamics?



The compressibility is essential for astrophysics fluid.

# 1. Introduction

Why do we need a numerical hydrodynamics?

Finite Difference

e.g., the 2nd order accurate finite difference of the 1st derivative,

$$\partial_x f \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

It is based on the Taylor expansion, i.e., the function f is differentiable.

⇒ Shock waves are the discontinuity of the physical variables.

The Taylor expansion is not applicable. ⇒ We need a numerical technique to handle the shock.

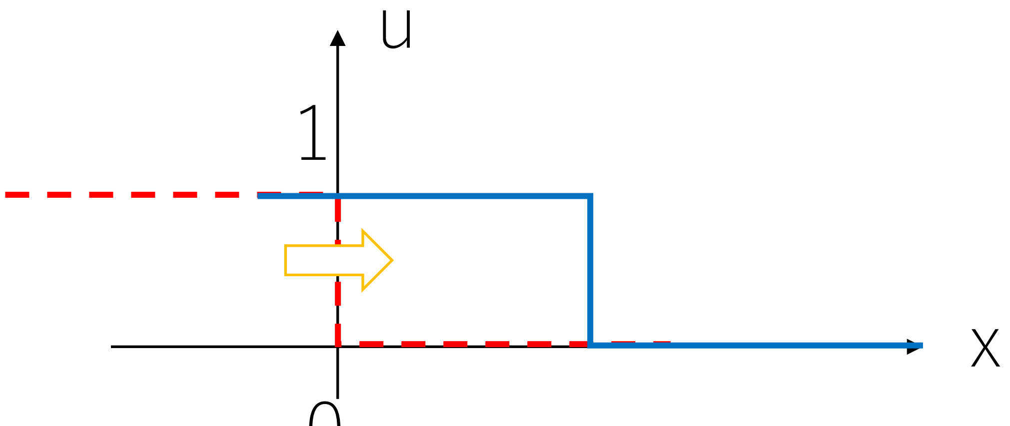
# 2. Finite Difference vs Finite Volume

One-dimensional advection equation with a constant speed

$$\partial_t u + a\partial_x u = 0,$$
$$a = \text{const.} > 0$$

The exact solution is $u = f(at - x)$

An initial profile advects in the positive x-direction with the speed a.

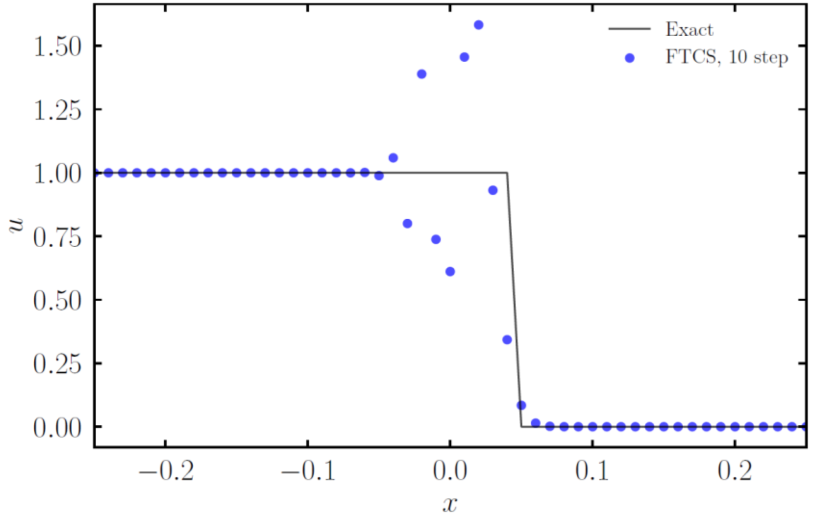



Initial profile ⇒ t > 0 profile

# 2. Finite Difference vs Finite Volume

Let's solver the equation numerically. We apply the 1st order in time-2nd order in space for the finite difference.

$$u_i^{n+1} = u_i^n + \frac{a\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^i\right)$$



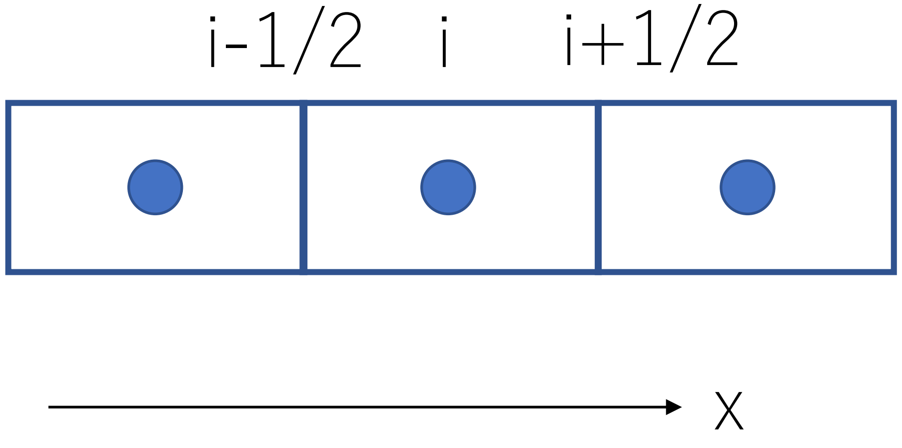

Von Neumann analysis reveals that the scheme is unstable for an arbitrary value of the Courant-Friedrich-Lewy number

$$\nu = a\frac{\Delta t}{\Delta x}$$

# 2. Finite Difference vs Finite Volume

Let's define a volume average by

$$\bar{u}_i \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x) dx$$




Then, we integrate the equation with Gauss's theorem;

$$\partial_t \bar{u}_i + \frac{\tilde{f}_{i+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2}}}{\Delta x} = 0,$$

$$\tilde{f}_{i\pm\frac{1}{2}} = (au)_{i\pm\frac{1}{2}}$$

The finite volume method relies on how to evaluate a numerical flux $\tilde{f}_{i\pm\frac{1}{2}}$

# 3. 1st-order up-wind scheme

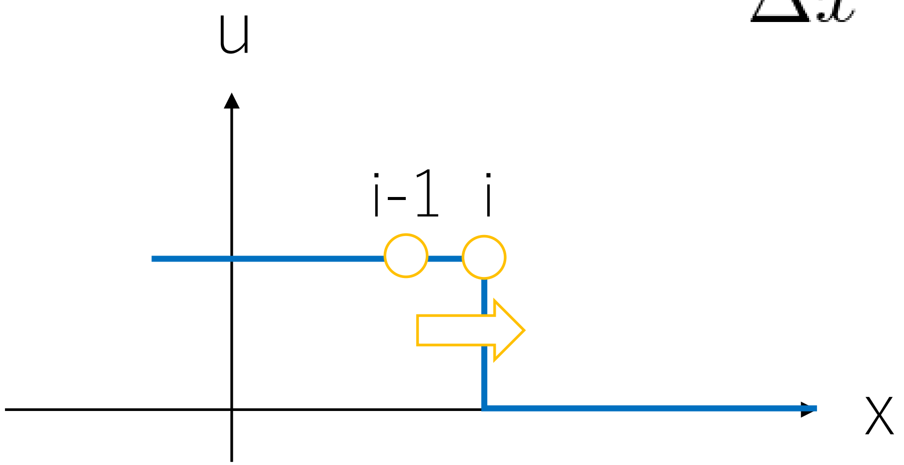
$$\tilde{f}_{i+\frac{1}{2}} = \frac{1}{2}\left[(f_{i+1} + f_i) - |a|\,(u_{i+1} - u_i)\right]$$

where we generalize to a < 0 case. For example, for a > 0

$$\tilde{f}_{i+\frac{1}{2}} = (a\bar{u})_i,\ \tilde{f}_{i-\frac{1}{2}} = (a\bar{u})_{i-1}$$

Therefore, the finite volume equation implies

$$\partial_t \bar{u}_i + \frac{(a\bar{u})_j - (a\bar{u})_{j-1}}{\Delta x} = 0$$



# 3. 1st-order up-wind scheme

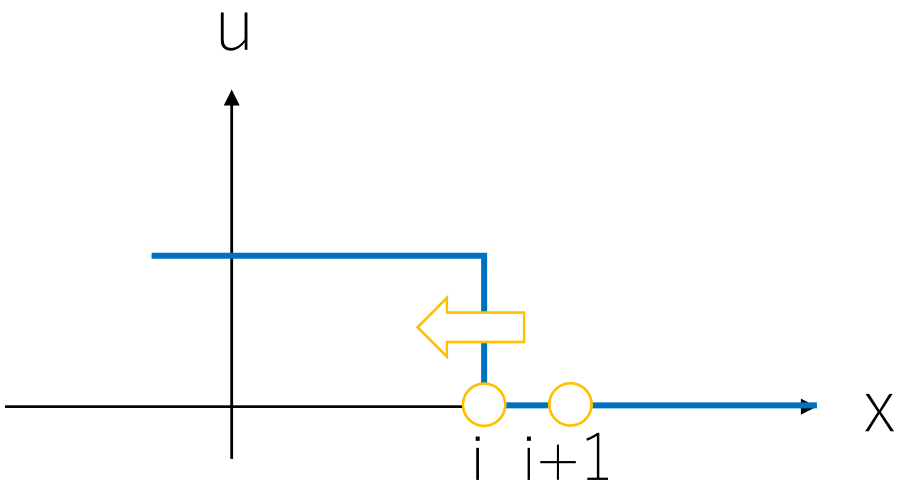
$$\tilde{f}_{i+\frac{1}{2}} = \frac{1}{2}\left[(f_{i+1} + f_i) - |a|\,(u_{i+1} - u_i)\right]$$

For a < 0

$$\tilde{f}_{i+\frac{1}{2}} = (a\bar{u})_{i+1},\ \tilde{f}_{i-\frac{1}{2}} = (a\bar{u})_i$$

Therefore, the finite volume equation implies

$$\partial_t \bar{u}_i + \frac{(a\bar{u})_{j+1} - (a\bar{u})_j}{\Delta x} = 0$$

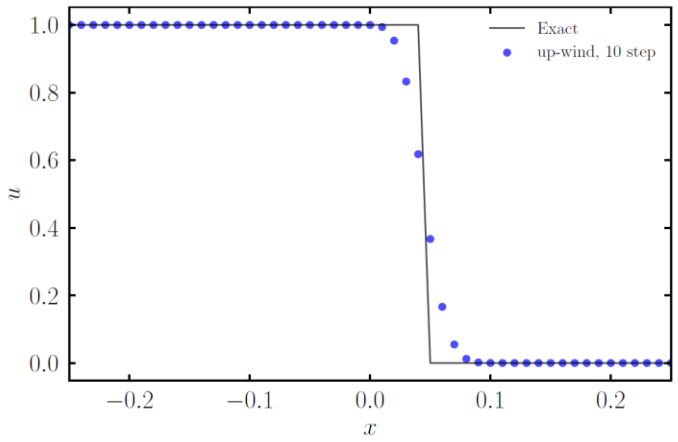



The 1st-order up-wind scheme takes into account from which direction the flow comes in.

⇒ Physically correct.

# 3. 1st-order up-wind scheme

Example




It correctly captures the advection without unphysical oscillations. But, the initial discontinuity is smeared out.

For example, for a > 0

$$\tilde{f}_{i-\frac{1}{2}} = (a\bar{u})_{i-1} = (a\bar{u})_i - \Delta x \partial_x f|_{x=x_i} + \frac{\Delta x^2}{2!}\partial_{xx} f|_{x=x_i} + O(\Delta x^3)$$

$$\partial_t \bar{u}_i + \partial_x f|_{x=x_i} + O(\Delta x) = 0$$

So, the task is to find a higher-order accurate scheme.

# 4. High Resolution Shock Capturing scheme

## Monotonicity preserving

If a function of u is monotonic in x for a given time $t^n$, it is also monotonic n x for the next time step $t^{n+1}$.

Any hydrodynamics schemes should guarantee the monotonicity preserving property.  Otherwise, unphysical oscillations will appear.

## Godunov's theorem

Any schemes higher than the 2nd order which has a following form,

$$u_i^{n+1} = \sum_{k=-k_L}^{k_R} b_k u_{i+k}^n,$$

cannot preserve the monotonicity where $b_k$ is constant and $k_{R/L}$ is non-negative integer.

# 4. High Resolution Shock Capturing scheme

<u>Proof</u>

$$\Delta u_i^{n+1} = u_{i+1}^{n+1} - u_i^{n+1} = \sum_{k=-k_L}^{k_R} b_k \Delta u_{i+k}^n$$

The monotonicity preserving means if $\Delta u^n_i > 0$, $\Delta u^{n+1}_i > 0$ should be satisfied for any value of i.

Let's consider the case in which $b_{k0} < 0$ for a particular value of $k_0$. Then, if $\Delta u^n_i = 0$ is satisfied except for $i = k_1$, for $i = -k_0 + k_1$,

$$\Delta u_{-k_0+k_1}^{n+1} = \sum_{k=-k_L}^{k_R} b_k \Delta u_{-k_0+k_1+k}^n = b_{k_0} \Delta u_{k_1}^n$$

$\Delta u^{n+1}_{-k0+k1}$ has a sigh opposite to $\Delta u^n_{k1}$.

⇒ Violating the monotonicity preserving ⇒ $b_k \geq 0$ for all the value of k

# 4. High Resolution Shock Capturing scheme

## Proof (cont.)

Truncation error for the scheme of the form

$$u_i^{n+1} = \sum_{k=-k_L}^{k_R} b_k u_{i+k}^n,$$

$$u_i^{n+1} - \sum_k b_k u_{i+k}^n = \sum_{l=0} (\Delta t)^l \frac{u_{tl}}{l!} - \sum_k b_k \sum_{l=0} (k\Delta x)^l \frac{u_{xl}}{l!}$$

$$= \sum_{l=0} \frac{u_{xl}}{l!} \left[ (\Delta t)^l (-a)^l - \sum_k b_k (k\Delta x)^l \right]$$

The truncation error disappears if

$$s_l \equiv \sum_k k^l b_k = (-\nu)^l$$

# 4. High Resolution Shock Capturing scheme

## Proof (cont.)

Therefore, the p-th order accurate in space and time if and only if the following conditions are satisfied

$$s_q \equiv \sum_k k^q b_k = (-\nu)^q$$

where $0 \leq q \leq p$ and $\nu = a\Delta t/\Delta x$. Let's consider the 2nd order accurate scheme;

$$s_0 = 1,\ s_1 = -\nu,\ s_2 = \nu^2$$

However,

$$s_2 = \sum_k k^2 b_k = \sum_k (k+\nu)^2 b_k - 2\nu \sum_k k b_k - \nu^2 \sum_k b_k$$

$$= \sum_k (k+\nu)^2 b_k - 2\nu s_1 - \nu^2 s_0 = \sum_k (k+\nu)^2 b_k + \nu^2 \geq \nu^2$$

# 4. High Resolution Shock Capturing scheme

Proof (cont.)

The equality of the last expression holds only for useless case;

$b_k=0$ for all the values of k or $\nu$ is an integer.

⇒ Godunov's theorem shows that there are no monotonicity-preserving linear scheme of the form

$$u_i^{n+1} = \sum_{k=-k_L}^{k_R} b_k u_{i+k}^n,$$

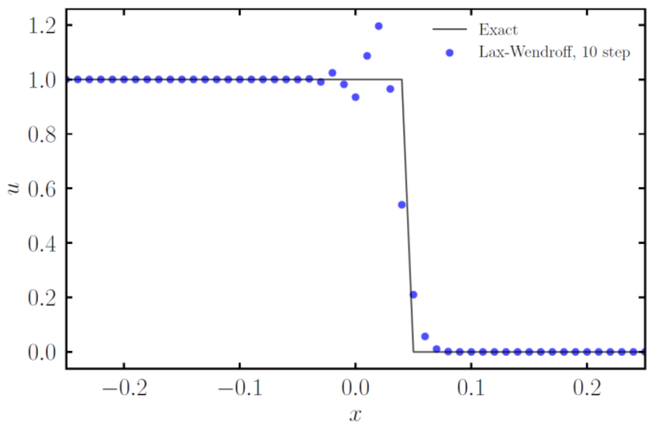
that are second or higher-order accurate.

Lax-Wendroff scheme (2nd-order in time and space)

$$u_{i+1}^n = u_i^n - \frac{\nu}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\nu^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

# 4. High Resolution Shock Capturing scheme

Example.




Therefore, to construct a higher-order scheme, we need a non-linear scheme. Let's consider the 1st-order up-wind scheme and the 2nd-order Lax-Wendroff scheme:

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \nu\left(\tilde{f}_{i+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2}}\right)$$

$$\tilde{f}_{i+\frac{1}{2}} = \bar{u}_i^n \text{ (up-wind)}$$

$$\tilde{f}_{i+\frac{1}{2}} = \bar{u}_i^n + \frac{1-\nu}{2}\left(\bar{u}_{i+1}^n - \bar{u}_i^n\right) \text{ (Lax-Wendroff)}$$

$$\text{for } \nu \geq 0$$

# 4. High Resolution Shock Capturing scheme

Similarly,

$$\tilde{f}_{i+\frac{1}{2}} = \bar{u}^n_{i+1} \text{ (up-wind)}$$

$$\tilde{f}_{i+\frac{1}{2}} = \bar{u}^n_{i+1} - \frac{1+\nu}{2}\left(\bar{u}^n_{i+1} - \bar{u}^n_i\right) \text{ (Lax-Wendroff)}$$

for $\nu \leq 0$

Introducing a flux limiter function $B_{i+1/2}$ such that the Lax-Wendroff scheme is reduced to be the up-wind scheme at the discontinuity.

$$\tilde{f}_{i+\frac{1}{2}} = \bar{u}^n_i + \frac{1-\nu}{2} B_{i+\frac{1}{2}}\left(\bar{u}^n_{i+1} - \bar{u}^n_i\right) \text{ for } \nu \geq 0$$

$$\tilde{f}_{i+\frac{1}{2}} = \bar{u}^n_{i+1} - \frac{1+\nu}{2} B_{i+\frac{1}{2}}\left(\bar{u}^n_{i+1} - \bar{u}^n_i\right) \text{ for } \nu \leq 0$$

$B_{i+1/2}$ is a function of $u^n_{i+k}$ with $k = 0, \pm 1, \pm 2$, .,.

⇒ The scheme is nonlinear in $u^n_i$ (Circumventing Godunov's theorem).

# 4. High Resolution Shock Capturing scheme

Then,
$$\bar{u}_i^{n+1} - \bar{u}_i^n = -\nu\Big[\bar{u}_i^n - \bar{u}_{i-1}^n + \frac{1-\nu}{2} B_{i+\frac{1}{2}}\left(\bar{u}_{i+1}^n - \bar{u}_i^n\right) - \frac{1-\nu}{2} B_{i-\frac{1}{2}}\left(\bar{u}_i^n - \bar{u}_{i-1}^n\right)\Big] \text{ for } \nu \geq 0$$

$$\bar{u}_i^{n+1} - \bar{u}_i^n = -\nu\Big[\bar{u}_{i+1}^n - \bar{u}_i^n - \frac{1+\nu}{2} B_{i+\frac{1}{2}}\left(\bar{u}_{i+1}^n - \bar{u}_i^n\right) + \frac{1+\nu}{2} B_{i-\frac{1}{2}}\left(\bar{u}_i^n - \bar{u}_{i-1}^n\right)\Big] \text{ for } \nu \leq 0$$

Finally,
$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\bar{u}_{i-1}^n - \bar{u}_i^n} = \nu\Big[1 + \frac{1-\nu}{2r_i} B_{i+\frac{1}{2}} - \frac{1-\nu}{2} B_{i-\frac{1}{2}}\Big] \text{ for } \nu \geq 0$$

$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\bar{u}_{i+1}^n - \bar{u}_i^n} = -\nu\Big[1 - \frac{1+\nu}{2} B_{i+\frac{1}{2}} + \frac{1+\nu}{2} r_i B_{i-\frac{1}{2}}\Big] \text{ for } \nu \leq 0$$

where

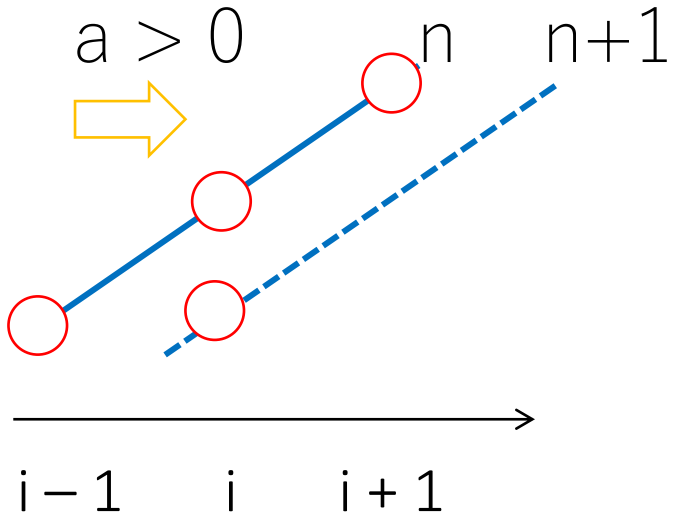
$$r_i = \frac{\Delta_{i-1}}{\Delta_i},\ \Delta_i = \bar{u}_{i+1}^n - \bar{u}_i^n$$

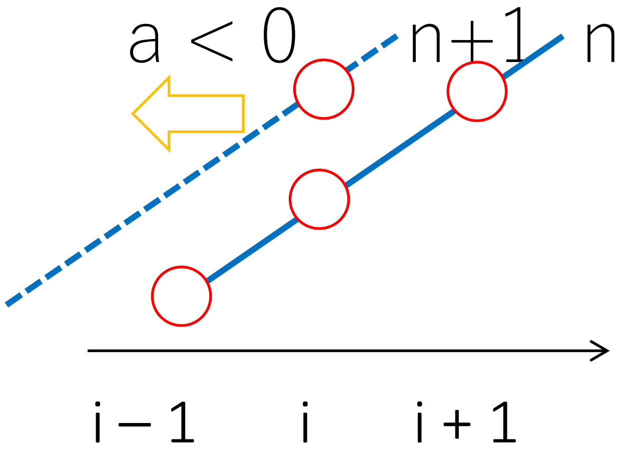
# 4. High Resolution Shock Capturing scheme

A sufficient condition to suppress numerical oscillation is

$$0 < \frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\bar{u}_{i-1}^n - \bar{u}_i^n} < 1 \text{ for } \nu \geq 0$$

$$0 < \frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\bar{u}_{i+1}^n - \bar{u}_i^n} < 1 \text{ for } \nu \leq 0$$



# 4. High Resolution Shock Capturing scheme

This condition yields

$$\frac{2}{1-\nu} \geq B_{i-\frac{1}{2}} - \frac{B_{i+\frac{1}{2}}}{r_i} \geq -\frac{2}{\nu} \text{ for } \nu \geq 0$$
$$\frac{2}{1+\nu} \geq B_{i+\frac{1}{2}} - B_{i-\frac{1}{2}} r_i \geq \frac{2}{\nu} \text{ for } \nu \leq 0$$

Further simplification is possible if we only consider a sufficient condition with $0 \leq |\nu| \leq 1$.

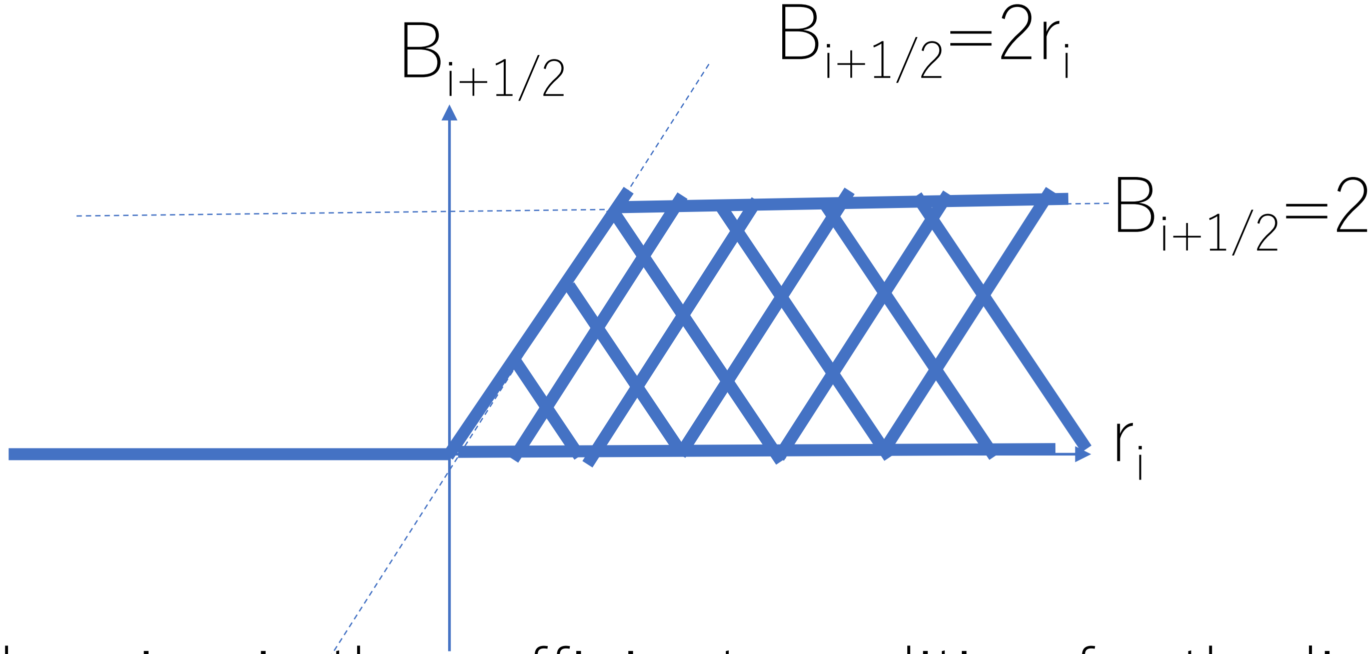
$$2 \geq B_{i-\frac{1}{2}} - \frac{B_{i+\frac{1}{2}}}{r_i} \geq -2 \text{ for } \nu \geq 0$$
$$2 \geq B_{i+\frac{1}{2}} - B_{i-\frac{1}{2}} r_i \geq -2 \text{ for } \nu \leq 0$$

This condition is automatically satisfied if

$$0 \leq B_{i+\frac{1}{2}} \leq 2,\ 0 \leq B_{i+\frac{1}{2}} \frac{\Delta_i}{\Delta_{i-\mathrm{sign}(\nu)}} \leq 2$$

# 4. High Resolution Shock Capturing scheme

$$0 \leq B_{i+\frac{1}{2}} \leq 2,\ 0 \leq B_{i+\frac{1}{2}} \frac{\Delta_i}{\Delta_{i-\mathrm{sign}(\nu)}} \leq 2$$



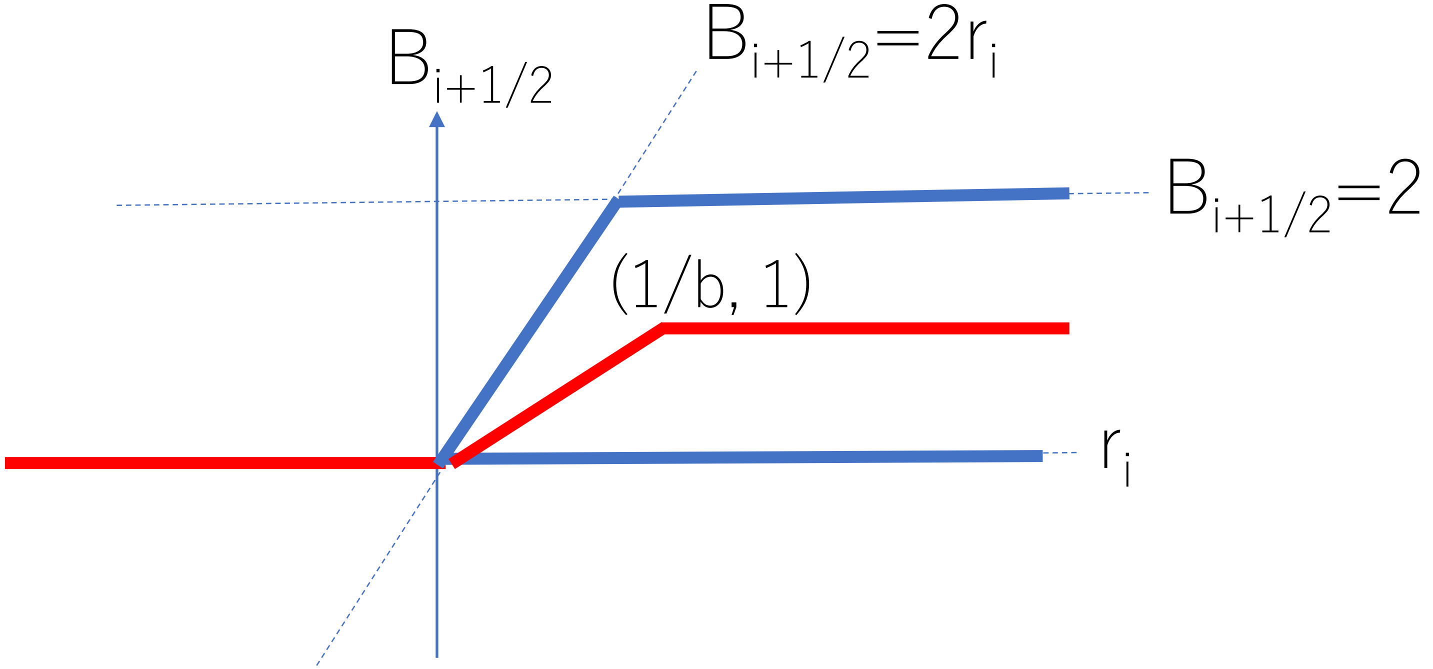

The hatched region is the sufficient condition for the limiter function of the monotonicity-preserving scheme

For $r_i < 0$, $B_{i+1/2} \Rightarrow 0$ (up-wind scheme)

# 4. High Resolution Shock Capturing scheme

One example of the flux limiter,

$$B_{i+1/2}(r_i)=\mathrm{minmod}(1,br_i)= \begin{cases} 1 \text{ when } |br_i|>1 \text{ and } br_i \geq 0, \\ br_i \text{ when } |br_i|<1 \text{ and } br_i \geq 0, \\ 0 \text{ when } br_i <0 \end{cases}$$



# 4. High Resolution Shock Capturing scheme

Total Variation Diminishing (TVD)

$$\mathrm{TV}\left(u^n\right) \equiv \sum_{i=-\infty}^{i=\infty} \left|u_{i+1}^n - u_i^n\right|$$

TVD condition is

$$\mathrm{TV}\left(u^n\right) \geq \mathrm{TV}\left(u^{n+1}\right)$$

Once a TVD scheme is employed, the monotonicity-preserving property is guaranteed.

# 4. High Resolution Shock Capturing scheme

Theorem

Suppose that a class of non-linear scheme is written as

$$u_i^{n+1} = u_i^n - C^n_{i-\frac{1}{2}}(u_i^n)\Delta_{i-1} + D^n_{i+\frac{1}{2}}(u_i^n)\Delta_i,$$

$$\text{where } \Delta_i = u_{i+1}^n - u_i^n.$$

Sufficient condition for scheme to be TVD is as follows:

$$C^n_{i+\frac{1}{2}} \geq 0,\ D^n_{i+\frac{1}{2}} \geq 0,\ 0 \leq C^n_{i+\frac{1}{2}} + D^n_{i+\frac{1}{2}} \leq 1$$

Proof

$$u_{i+1}^{n+1} - u_i^{n+1} = \left(u_{i+1}^n - u_i^n\right)\left(1 - C^n_{i+\frac{1}{2}} - D^n_{i+\frac{1}{2}}\right)$$

$$+ C^n_{i-\frac{1}{2}}\left(u_i^n - u_{i-1}^n\right) + D^n_{i+\frac{3}{2}}\left(u_{i+2}^n - u_{i+1}^n\right)$$

$$\Rightarrow |u_{i+1}^{n+1} - u_i^{n+1}| \leq \left|u_{i+1}^n - u_i^n\right|\left(1 - C^n_{i+\frac{1}{2}} - D^n_{i+\frac{1}{2}}\right)$$

$$+ C^n_{i-\frac{1}{2}}\left|u_i^n - u_{i-1}^n\right| + D^n_{i+\frac{3}{2}}\left|u_{i+2}^n - u_{i+1}^n\right|$$

# 4. High Resolution Shock Capturing scheme

Proof (cont.)

$$
\begin{aligned}
\mathrm{TV}(u^{n+1}) &= \sum_i |u_{i+1}^{n+1} - u_i^{n+1}| \\
&\leq \sum_i |u_{i+1}^{n} - u_i^{n}| \left(1 - C_{i+\frac{1}{2}}^n - D_{i+\frac{1}{2}}^n\right) \\
&+ \sum_i C_{i+\frac{1}{2}}^n \left|u_{i+1}^{n} - u_i^{n}\right| + \sum_i D_{i+\frac{1}{2}}^n \left|u_{i+1}^{n} - u_i^{n}\right| \\
&= \mathrm{TV}(u^n)
\end{aligned}
$$

Therefore, if we invent a higher-order scheme in the form of

$$
\begin{aligned}
&u_i^{n+1} = u_i^n - C_{i-\frac{1}{2}}^n(u_i^n)\Delta_{i-1} + D_{i+\frac{1}{2}}^n(u_i^n)\Delta_i, \\
&\text{where } \Delta_i = u_{i+1}^n - u_i^n.
\end{aligned}
$$

The scheme is TVD if $C_{i+\frac{1}{2}}^n \geq 0,\ D_{i+\frac{1}{2}}^n \geq 0,\ 0 \leq C_{i+\frac{1}{2}}^n + D_{i+\frac{1}{2}}^n \leq 1$

# 4. High Resolution Shock Capturing scheme

One example of HRSC scheme = M(onotone) U(pstream-centered) S(cheme) for C(onservation) L(aw)

<u>Volume averaged quantity</u>

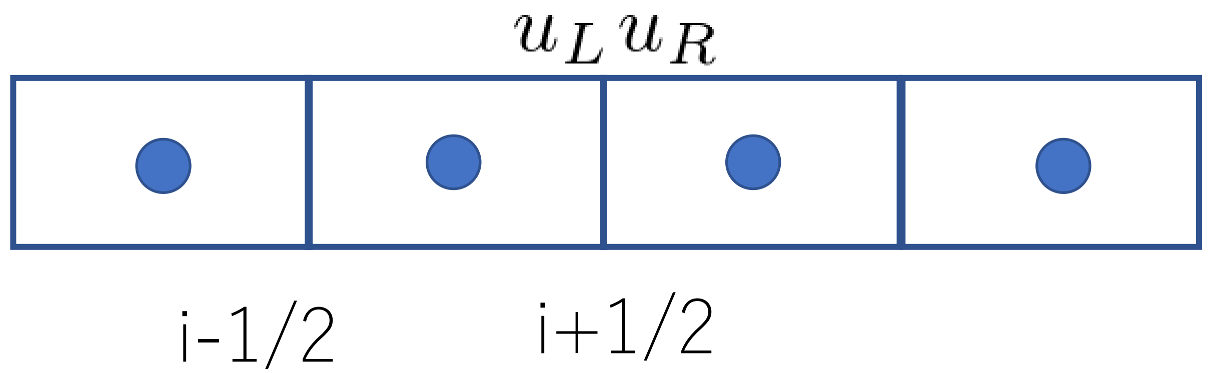
$$\bar{u}_i \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x) dx$$

<u>Piecewise parabolic cell reconstruction</u>

$$u(x) = \bar{u}_i + \frac{1}{\Delta x} (x - x_i) \, \partial_x u|_i + \frac{1}{2\Delta x^2} \left[ (x - x_i)^2 - \frac{\Delta x^2}{12} \right] \partial_{xx} u|_i$$

# 4. High Resolution Shock Capturing scheme

## MUSCL scheme




$$(u_L)_{i+\frac{1}{2}} = u_i + \frac{1}{6}\Phi(r^+_{i-1})\Delta_{i-1} + \frac{1}{3}\Phi(r^-_i)\Delta_i$$

$$(u_R)_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{3}\Phi(r^+_i)\Delta_i - \frac{1}{6}\Phi(r^-_{i+1})\Delta_{i+1}$$

$$\Delta_i = u_{i+1} - u_i,\; r^\pm_i = \frac{\Delta_{i\pm1}}{\Delta_i}$$

1. The first term is reduced to be the up-wind scheme for the non-smooth flow
2. The second and third terms are higher-order representation of the gradient
3. $\Phi$ is the flux-limiter, one choice is the minmod function

# 4. High Resolution Shock Capturing scheme

## MUSCL scheme

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\tilde{f}_{i+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2}}\right),$$

$$\tilde{f}_{i+\frac{1}{2}} = \frac{a}{2}\left[(u_R)_{i+\frac{1}{2}} + (u_L)_{i+\frac{1}{2}} - \frac{|a|}{a}\left((u_R)_{i+\frac{1}{2}} - (u_L)_{i+\frac{1}{2}}\right)\right]$$

For a > 0, $\tilde{f}_{i+\frac{1}{2}} = a\,(u_L)_{i+\frac{1}{2}}$

if the flow is not smooth, the 1st-order upwind scheme is recovered

$$\tilde{f}_{i+\frac{1}{2}} = au_i$$

For a < 0, $\tilde{f}_{i+\frac{1}{2}} = a\,(u_R)_{i+\frac{1}{2}}$

if the flow is not smooth, the 1st-order upwind scheme is recovered

$$\tilde{f}_{i+\frac{1}{2}} = au_{i+1}$$

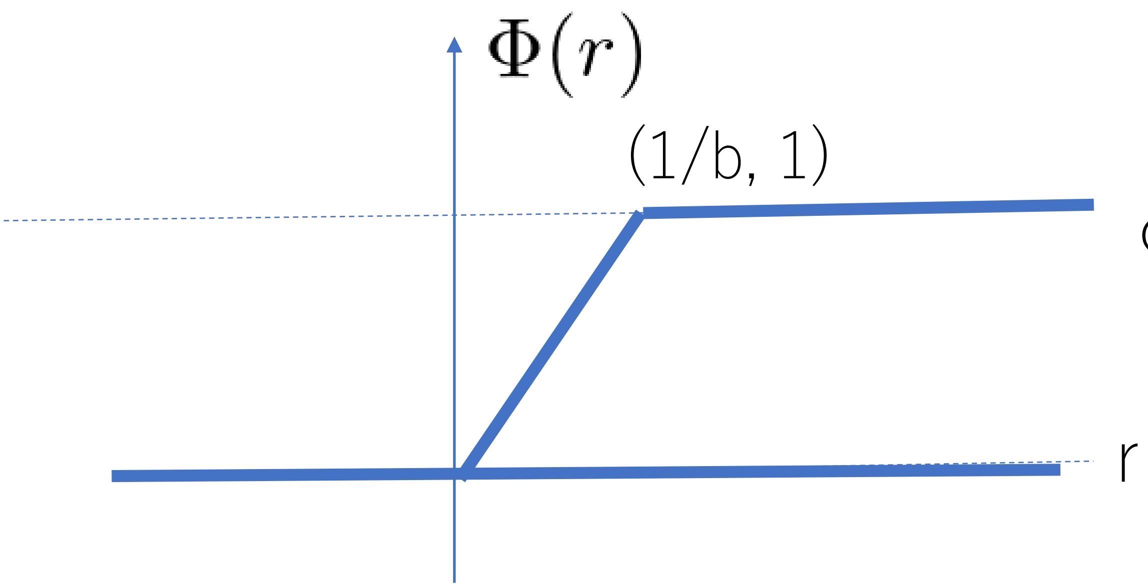
# 4. High Resolution Shock Capturing scheme

## Minmod function and compression parameter

$$(u_L)_{i+\frac{1}{2}} = u_i + \frac{1}{6}\Phi(r^+_{i-1})\Delta_{i-1} + \frac{1}{3}\Phi(r^-_i)\Delta_i$$

$$(u_R)_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{3}\Phi(r^+_i)\Delta_i - \frac{1}{6}\Phi(r^-_{i+1})\Delta_{i+1}$$

$$\Phi(r) = \mathrm{minmod}(1, br)$$




$\Phi$ =1 is desired for a wide range of r.
b should be large as much as possible.

# 4. High Resolution Shock Capturing scheme

<u>Compression parameter is determined by the TVD condition</u>

$$
\begin{aligned}
\bar{u}_i^{n+1} &= \bar{u}_i^n \\
&- \frac{\Delta t}{\Delta x} a^+ \left[ \Delta_{i-1} + \frac{1}{6}\Phi(r_{i-1}^+)\Delta_{i-1} + \frac{1}{3}\Phi(r_i^-)\Delta_i - \frac{1}{6}\Phi(r_{i-2}^+)\Delta_{i-2} - \frac{1}{3}\Phi(r_{i-1}^-)\Delta_{i-1} \right] \\
&- \frac{\Delta t}{\Delta x} a^- \left[ \Delta_i - \frac{1}{3}\Phi(r_i^+)\Delta_i - \frac{1}{6}\Phi(r_{i+1}^-)\Delta_{i+1} + \frac{1}{3}\Phi(r_{i-1}^+)\Delta_{i-1} + \frac{1}{6}\Phi(r_i^-)\Delta_i \right]
\end{aligned}
$$

where $a^\pm = \dfrac{1 \pm \mathrm{sign}(a)}{2} a$

$$
\Phi(r_i^\pm)\Delta_i = \mathrm{minmod}\left(1, b\frac{\Delta_{i\pm1}}{\Delta_i}\right)\Delta_i = \mathrm{minmod}\left(\frac{\Delta_i}{\Delta_{i\pm1}}, b\right)\Delta_{i\pm1}
$$

# 4. High Resolution Shock Capturing scheme

Compression parameter is determined by the TVD condition

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\bar{a}^+ \Delta_{i-1} + \bar{a}^- \Delta_i\right)$$

$$\bar{a}^+ = a^+ \left[1 + \frac{1}{6}\Phi(r_{i-1}^+) + \frac{1}{3}\text{minmod}\left(\frac{\Delta_i}{\Delta_{i-1}}, b\right) - \frac{1}{6}\text{minmod}\left(\frac{\Delta_{i-2}}{\Delta_{i-1}}, b\right) - \frac{1}{3}\Phi(r_{i-1}^-)\right]$$

$$\bar{a}^- = a^- \left[1 - \frac{1}{3}\Phi(r_i^+) - \frac{1}{6}\text{minmod}\left(\frac{\Delta_{i+1}}{\Delta_i}, b\right) + \frac{1}{3}\text{minmod}\left(\frac{\Delta_{i-1}}{\Delta_i}, b\right) + \frac{1}{6}\Phi(r_i^-)\right]$$

In Harten's theorem

$$u_i^{n+1} = u_i^n - C_{i-\frac{1}{2}}^n(u_i^n)\Delta_{i-1} + D_{i+\frac{1}{2}}^n(u_i^n)\Delta_i,$$

where $\Delta_i = u_{i+1}^n - u_i^n$.

$$C_{i+\frac{1}{2}}^n \geq 0,\ D_{i+\frac{1}{2}}^n \geq 0,\ 0 \leq C_{i+\frac{1}{2}}^n + D_{i+\frac{1}{2}}^n \leq 1$$

# 4. High Resolution Shock Capturing scheme

Compression parameter is determined by the TVD condition

$$\bar{a}^+ \geq 0,\ \bar{a}^- \leq 0,\ 0 \leq \bar{a}^+ - \bar{a}^- \leq \frac{\Delta x}{\Delta t}$$

When

$$\Delta_i \Delta_{i-1} < 0,\ \frac{\Delta_{i-2}}{\Delta_{i-1}} > b$$

the first condition is the strongest and it gives

$$1 \leq b \leq 4$$

When

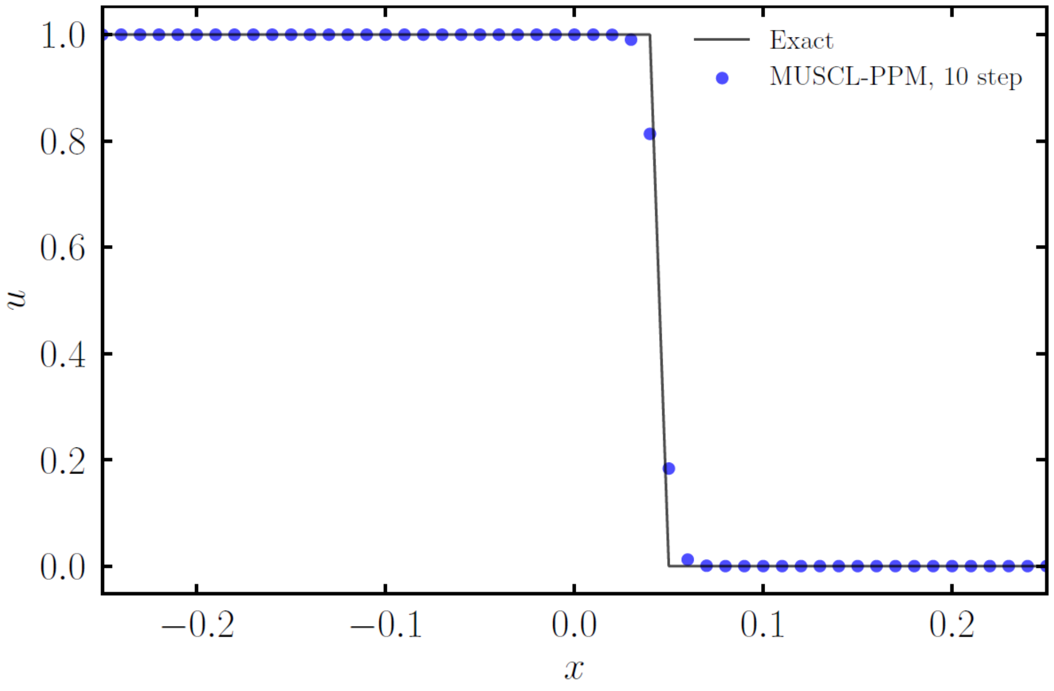
$$\Delta_i \Delta_{i-1} < 0,\ \frac{\Delta_{i+1}}{\Delta_i} > b$$

the 2nd condition is the strongest and it gives

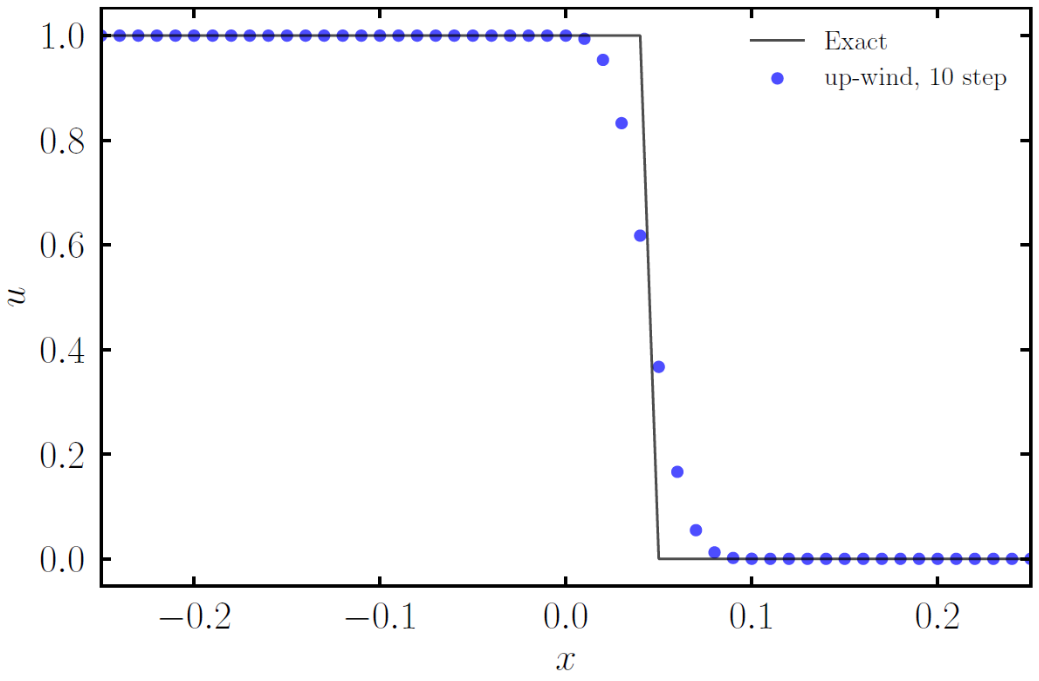
$$1 \leq b \leq 4$$

# 4. High Resolution Shock Capturing scheme

MUSCL-PPM




1st-order up-wind




The discontinuity is correctly and sharply captured.

# 5. 1D Berger equation and characteristic curve

1D advection equation: $f(u) = au$

Characteristic speed = a = constant

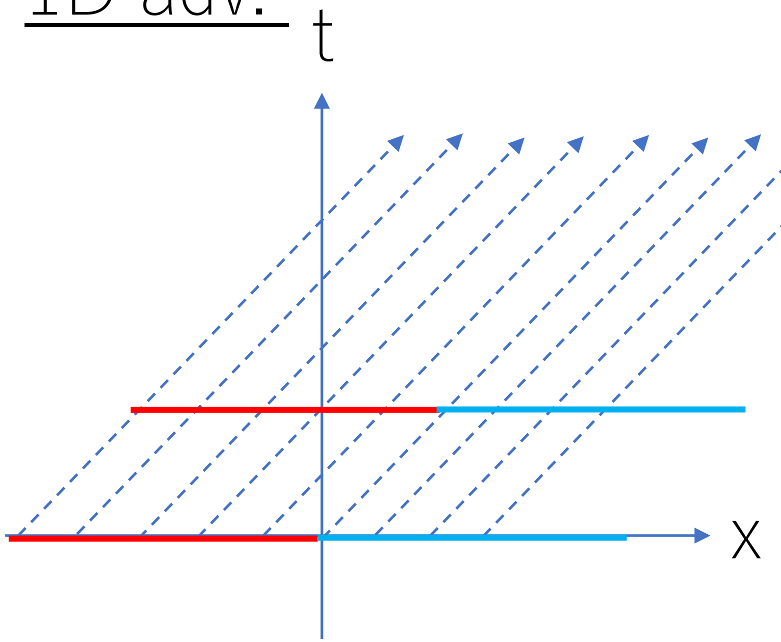
Characteristic curve: $dx/dt = a$

1D inviscid Berger equation: $f(u)=u^2/2$

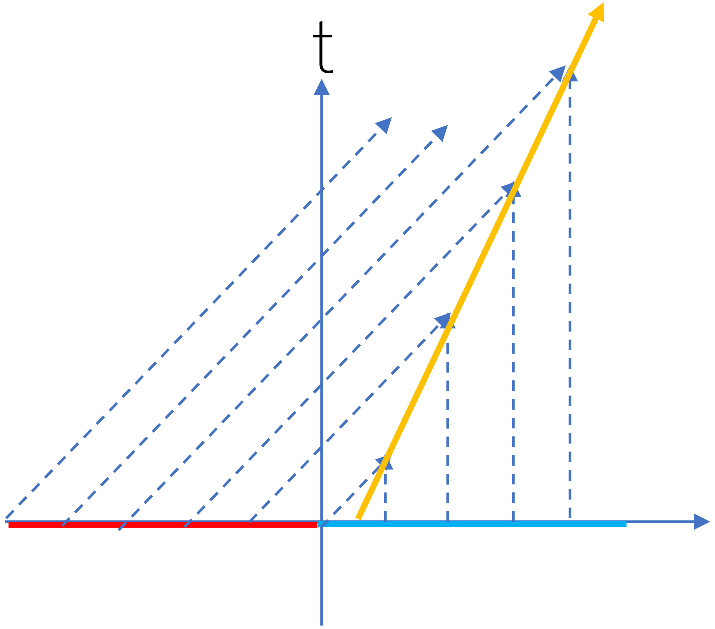
Characteristic speed = $\partial f / \partial u = u$

Characteristic curve: $dx/dt = u$

1D adv.




1D Berger eq.




$u(0,x)=1$ $(x<0)$

$u(0,x)=0$ $(x>0)$

Shock wave appears.

# Short summary of the 1st course

1. Finite volume method and 1st-order up-wind scheme is essential for Numerical Hydrodynamics. Caveat: Diffusive

2. Godunov's theorem says there is no monotonicity-preserving linear scheme which is 2nd order or higher accurate.

⇒ Necessary to invent a monotonicity-preserving non-linear scheme
One example = Introducing flux limiter

3. MUSCL scheme is one example of the higher-order monotonicity preserving scheme.

Now, we know how to handle the scalar equation which may contain a discontinuity.

## Contents

# 6. Euler equation and characteristic speed

## Euler equation

$$\partial_t \vec{U} + \partial_x \vec{F} = 0,$$

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \ \vec{F} = \begin{pmatrix} u \\ \rho u^2 + P \\ (e+p)u \end{pmatrix}$$

A solution may contain a shock wave where we can’t define differential.

## Weak solution

Suppose that we have a support w(x,t) which has a finite value in a limited region of the spacetime and infinitely differentiable, U is a solution if

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \left(\partial_t \vec{U} + \partial_x \vec{F}\right) w(x,t) = 0$$

is satisfied for an arbitrary w.

# 6. Euler equation and characteristic speed

By performing the partial integral,

$$-\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \left(\vec{U}\partial_t w + \vec{F}\partial_x w\right) = 0$$

Let`s consider i component for U and F and suppose $U^i$ has an initial condition;

$$U^i(x,0) = u_1(x<0),\ u_2(x>0)$$

Suppose that a characteristic speed $\lambda$ (will discuss later on) is constant, we get a solution

$$U^i(x,t) = u_1(x-\lambda t<0;\ \text{region A}),\ u_2(x-\lambda t>0;\ \text{region B})$$

In each region, this solution satisfies the original solution, but it is not along the characteristic curve.

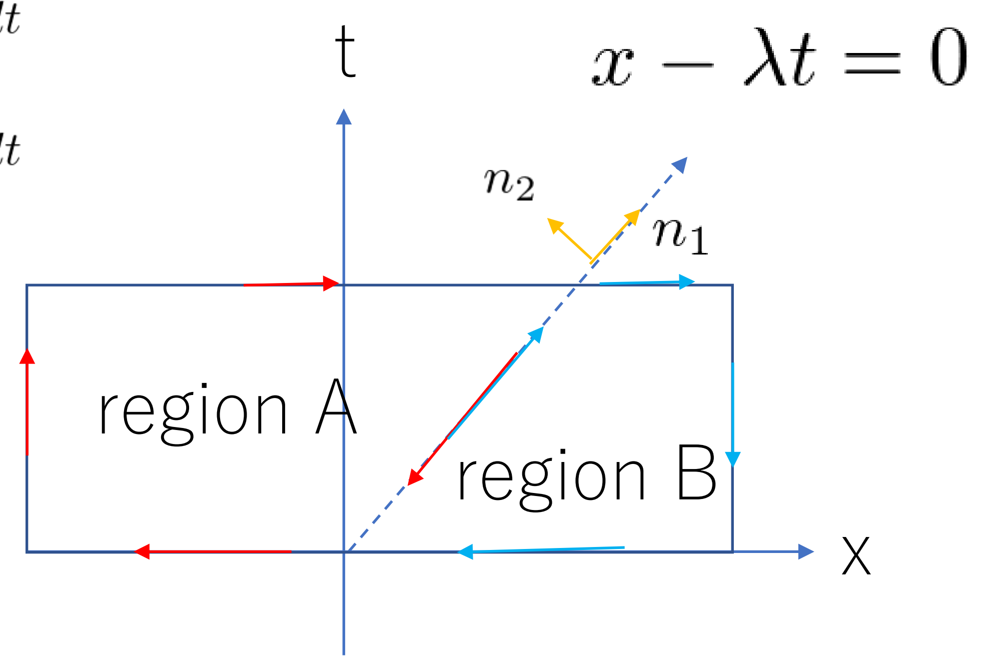
# 6. Euler equation and characteristic speed

If we plug this solution in

$$-\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \left(\vec{U}\partial_t w + \vec{F}\partial_x w\right) = 0$$

and consider a closed region (see figure), we can integrate along the line.

$$-\int\int_{A+B} \left(u\partial_t w + f\partial_x\right) dxdt = -u_1 \int\int_A \partial_t w dxdt - u_2 \int\int_B \partial_t w dxdt$$
$$-f(u_1)\int\int_A \partial_x w dxdt - f(u_2)\int\int_B \partial_x w dxdt$$




From Stokes's theorem, the 1st term is

$$-u_1 \int\int_A \partial_t w dxdt = -u_1 \int_{C} (-w) n_1 ds$$

and the 2nd term is

$$-u_2 \int\int_B \partial_t w dxdt = -u_2 \int_{C} (-w)(-n_1) ds$$

# 6. Euler equation and characteristic speed

Similarly, the 3rd and 4th terms are

$$-f(u_1)\int\int_A \partial_x w dx dt = -f(u_1)\int\int w n_2 ds$$
$$-f(u_2)\int\int_B \partial_x w dx dt = +f(u_1)\int\int w n_2 ds$$

Then,

$$(\lambda[u]-[f])\int\int w n_2 ds = 0,$$

where $[u]=u_1-u_2,\ [f]=f(u_1)-f(u_2)$ and we use $n_1=\lambda n_2$

Therefore, to hold this equation for an arbitrary w,

$$\lambda[u]-[f]=0$$

If we repeat the same procedure for the other components in U and F,

$$-\lambda[U]+[F]=0$$

# 6. Euler equation and characteristic speed

Rankine-Hugoniot relation (jump condition)

$$-\lambda\,[U] + [F] = 0$$

For the characteristic speed $\lambda$, we consider the non-conservative form of the Euler equation;

$$\partial_t \left( \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right) + M\partial_x \left( \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right) = 0,$$

where

$$M = \frac{\partial F}{\partial Q} = \frac{\partial(f_1, f_2, f_3)}{\partial(q_1, q_2, q_3)}$$

The eigen value of A is the characteristic speed $\lambda$ of the simple wave

$$\frac{dx}{dt} = \lambda_k,\ (k = 1, 2, 3)$$

# 6. Euler equation and characteristic speed

Let's parameterize the group of the characteristic curves with $\lambda_k$ as

$$\phi_k(x,t) = \xi$$

Since $\phi_k$ is constant along the characteristics,

$$0 = d\phi_k \Rightarrow \partial_t \phi_k + \lambda_k \partial_x \phi_k = 0$$

For a simple wave $Q_k$ with $\lambda_k$

$$\partial_t Q_k + \lambda_k \partial_x Q_k = 0 \Rightarrow dQ_k = 0 \text{ along the characteristic.}$$

Therefore,

$$Q_k = Q_k(\phi_k)$$

# 6. Euler equation and characteristic speed

The Euler equation is

$$\partial_t \phi_k \frac{\partial Q_k}{\partial \phi_k} + M \partial_x \phi_k \frac{\partial Q_k}{\partial \phi_k} = \left(\partial_t \phi_k + M \partial_x \phi_k\right) \frac{\partial Q_k}{\partial \phi_k} = 0 \Rightarrow \left(M - \lambda_k I\right) \partial_x \phi_k \frac{\partial Q_k}{\partial \phi_k} = 0$$

Therefore

$$\frac{\partial Q_k}{\partial \phi_k} \propto R_k,$$

where $R_k$ is the right eigen vector of $M$.

This gives us a constant when we across the characteristic (implying increasing $\phi_k$ direction) by

$$dQ_k - R_k d\phi_k = 0 \ \Rightarrow \int \left(dQ_k - R_k d\phi_k\right) = \text{const.}$$

It is called the generalized Riemann invariants.

# 6. Euler equation and characteristic speed

Also if we use the left eigen vector $L_k$ of the matrix M with $\lambda_k$

$$0 = L_k(\partial_t Q_k + M\partial_x Q_k) = L_k(\partial_t Q_k + \lambda_k \partial_x Q_k)$$

If we define the differential $d\sigma$ along the characteristic $\phi_k(x,t) = \xi$

$$L_k\left(\frac{dQ_k}{d\sigma}\right) = L_k\left(\partial_t Q_k \frac{dt}{d\sigma} + \partial_x Q_k \frac{dx}{d\sigma}\right) = L_k\left(\partial_t Q_k + \lambda_k \partial_x Q_k\right)\frac{dt}{d\sigma} = 0$$

Therefore,

$$\int L_k dQ = \text{constant}$$

along the characteristics. It is called the Riemann invariant.

# 6. Euler equation and characteristic speed

The characteristic speed and the eigen vectors of the Euler equation are calculated by

$$\lambda_1 = u - c_s,\ \lambda_2 = u,\ \lambda_3 = u + c_s,$$

$$L_1 = \left(0, 1, -\frac{1}{\rho c_s}\right),\ L_2 = \left(1, 0, -\frac{1}{c_s^2}\right),\ L_3 = \left(0, 1, \frac{1}{\rho c_s}\right),$$

$$R_1 = \begin{pmatrix} 1 \\ -\frac{c_s}{\rho} \\ c_s^2 \end{pmatrix},\ R_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ R_3 = \begin{pmatrix} 1 \\ \frac{c_s}{\rho} \\ c_s^2 \end{pmatrix},$$

For $\lambda_1$, the generalized Riemann invariant is

$$dQ_1 = \begin{pmatrix} d\rho \\ du \\ dp \end{pmatrix} = R_1 d\phi_1 = \begin{pmatrix} 1 \\ -\frac{c_s}{\rho} \\ c_s^2 \end{pmatrix}$$

# 6. Euler equation and characteristic speed

It is reduced to

$$du + \frac{c_s}{\rho} d\rho = 0,$$
$$dp - c_s^2 d\rho = 0$$

Then, if we assume $\gamma$-law EOS $p=(\gamma-1)\rho\varepsilon$, it is integrated by

$$u + \frac{2c_s}{\gamma - 1} = \mathrm{constant},$$
$$\frac{P}{\rho^\gamma} = \mathrm{constant}$$

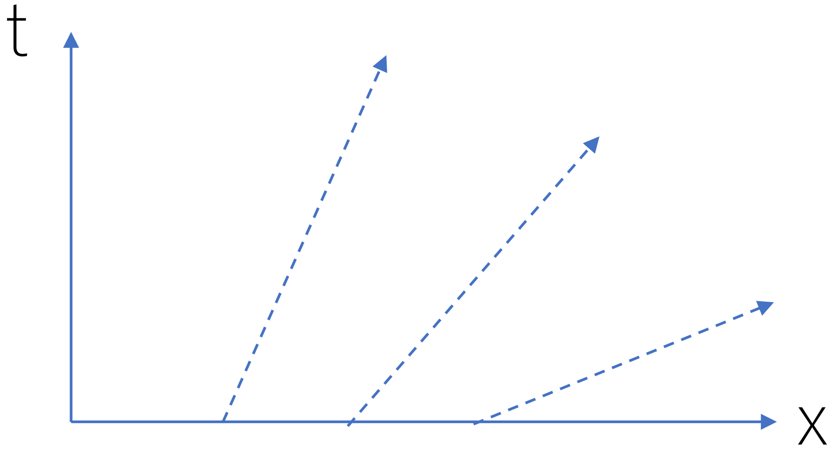
The entropy is conserved when we across the characteristic with $\lambda_1$. From the definition of the sound wave

$$dc_s = \frac{1}{2c_s}\left(\frac{\gamma}{\rho}dp - \frac{\gamma p}{\rho^2}\right) = \frac{1-\gamma}{2}du \Rightarrow d(u - c_s) = \frac{1+\gamma}{2}du$$

# 6. Euler equation and characteristic speed

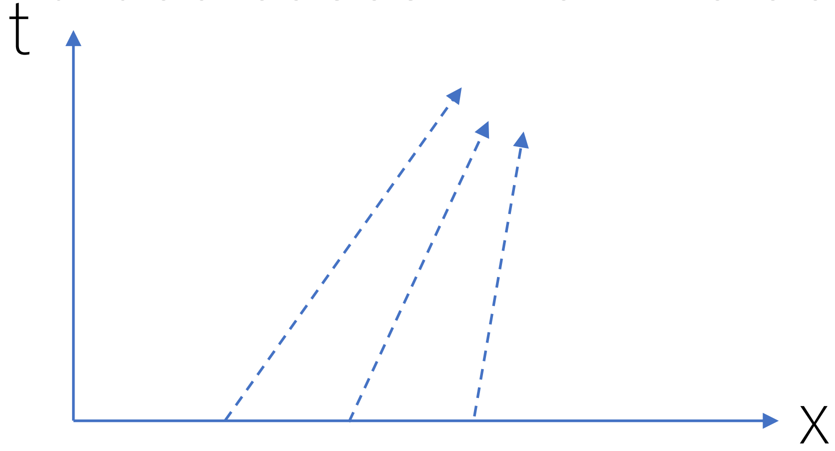
$$d(u - c_s) = \frac{1+\gamma}{2} du$$

If u increases when we across the characteristics, $\lambda_1$ also increases.




It presents the expansion wave (rarefaction wave)

If u decreases when we across the characteristics, $\lambda_1$ also decreases.




It present the compression wave (shock wave).

# 6. Euler equation and characteristic speed

Similarly, for the characteristics with $\lambda_2$, the generalized Riemann invariant is

$$dQ_2 = \begin{pmatrix} d\rho \\ du \\ dp \end{pmatrix} = R_2 d\phi_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow du = 0,\ dp = 0$$

Therefore, the velocity and pressure do not change when we across the characteristic. It presents the contact discontinuity.

Also, the Riemann invariant is

$$L_2 dQ_2 = d\rho - \frac{dp}{c_s^2} \Rightarrow \ln\left(\frac{P}{\rho^\gamma}\right) = \mathrm{constant}$$

Therefore, the entropy is constant along the characteristics. It is called the entropy wave.

# 6. Euler equation and characteristic speed

Similarly, for the characteristics with $\lambda_3$, the generalized Riemann invariant is

$$u - \frac{2c_s}{\gamma - 1} = \text{constant}, \quad \frac{P}{\rho^\gamma} = \text{constant}$$

$$d(u + c_s) = \frac{\gamma + 1}{2} du$$

It present the expansion/compression wave.

<u>Short summary</u>

1. Weak solution of the Euler equation results in the Rankine-Hugoniot Relation

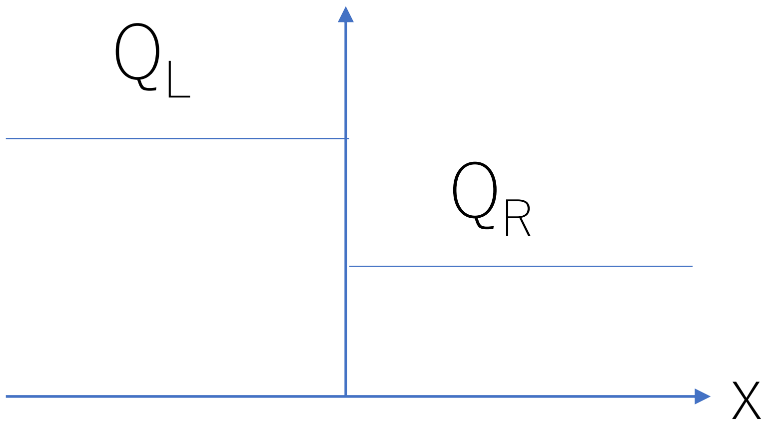
2. Three characteristic waves appear; two expansion/compression waves and contact discontinuity

3. The Riemann/generalized Riemann invariants exists along/when we across the characteristics.

# 6. Euler equation and characteristic speed

Solving the Riemann problem analytically. Let's consider a specific initial condition with

$$U_L = \begin{pmatrix} \rho_1 \\ u_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; U_R = \begin{pmatrix} \rho_4 \\ u_4 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0 \\ 0.1 \end{pmatrix}$$



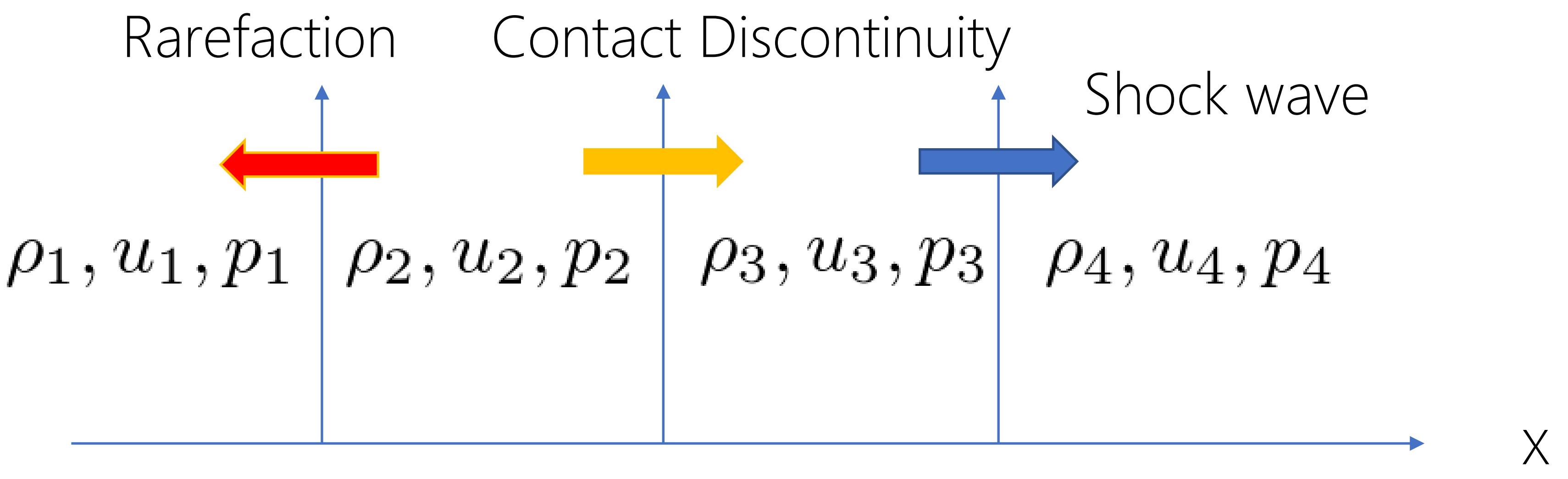

With $\gamma = 1.4$.

This problem is called Sod's problem. The sound speed is $c_{sL} = 1.4$, $c_{sR} = 1.4 \times 0.8$.

The wave structure is left-propagating rarefaction wave, the contact discontinuity, and the right-propagating shock.

# 6. Euler equation and characteristic speed

The wave structure is left-propagating rarefaction wave, the contact discontinuity, and the right-propagating shock.




The task is to calculate the intermediate state (2,3) for a given (1,4) sate.

# 6. Euler equation and characteristic speed

Let's consider the Rankine-Hugoniot relation with shock wave-comoving frame ( $\lambda$ =0), [F]=0.

$$\rho_4 u_4 - \rho_3 u_3 = 0,$$
$$\rho_4 u_4^2 + p_4 - \rho_3 u_3^2 - p_3 = 0,$$
$$(e_4 + p_4) u_4 - (e_3 + p_3) u_3 = 0$$

From the first two equations,

$$u_4^2 - u_3^2 = \frac{p_3}{\rho_4} + \frac{p_3}{\rho_3} - \frac{p_4}{\rho_3} - \frac{p_4}{\rho_4}$$

From the third equation with the $\gamma$ -law EOS,

$$u_4^2 - u_3^2 = -\frac{2\gamma}{\gamma - 1}\left(\frac{p_4}{\rho_4} - \frac{p_3}{\rho_3}\right)$$

# 6. Euler equation and characteristic speed

Then,

$$\left(\frac{\rho_3}{\rho_4}-\frac{\gamma+1}{\gamma-1}\right)\left(\frac{p_3}{p_4}+\frac{\gamma+1}{\gamma-1}\right)=-\frac{2\gamma}{(\gamma-1)^2}$$

Also,

$$\frac{p_3}{p_4}-1=\gamma\tilde{M}_4^2\left(1-\frac{\rho_4}{\rho_3}\right),\ \tilde{M}_4=\frac{u_4}{c_{s4}}$$

We can solve the quadratic equation for $p_3/p_4$ by

$$\frac{p_3}{p_4}=1+\frac{2\gamma}{\gamma+1}\left(\tilde{M}_4^2-1\right)$$

If we go back to the inertia frame,

$$\frac{p_3}{p_4}=1+\frac{2\gamma}{\gamma+1}\left(M_4^2-1\right),\ M_4=\frac{\lambda}{c_{s4}}$$

# 6. Euler equation and characteristic speed

Then,

$$\lambda = c_{s4}\left(\frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma}\frac{p_3}{p_4}\right)^{1/2}$$

From the Rankine-Hugoniot relation,

$$-\lambda(\rho_4 - \rho_3) - \rho_3 u_3 = 0 \;\Rightarrow u_3 = \lambda\frac{2}{\gamma+1}\frac{M_4^2-1}{M_4^2}$$

If we plug $\lambda$ and $M_4$ in this equation, we get

$$u_3 = c_{s4}\left(\frac{p_3}{p_4} - 1\right)\left(\frac{\frac{2}{\gamma}}{(\gamma+1)\frac{p_3}{p_4} + \gamma - 1}\right)^{1/2}$$

# 6. Euler equation and characteristic speed

Next, we consider the rarefaction wave (1,2). The generalized Riemann invariants are

$$\frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}, \quad \frac{c_{s1}}{\gamma - 1} = u_2 + \frac{2c_{s2}}{\gamma - 1},$$

$$\Rightarrow u_2 = \frac{2c_{s1}}{\gamma - 1}\left(1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{2\gamma}}\right)$$

Finally, if we consider the contact discontinuity, the velocity and pressure should be constant: $u_2 = u_3,\ p_2 = p_3$

Then, we obtain

$$c_{s4}\left(\frac{p_3}{p_4} - 1\right)\left(\frac{\frac{2}{\gamma}}{(\gamma + 1)\frac{p_3}{p_4} + \gamma - 1}\right)^{1/2} = \frac{2c_{s1}}{\gamma - 1}\left(1 - \left(\frac{p_3}{p_1}\right)^{\frac{\gamma-1}{2\gamma}}\right)$$

# 7. Roe's method

Sod's problem is a one example of the Godunov method (exact solution of the Riemann problem). But, it is infeasible to solve the Riemann problem from the computational point of view.

We need to seek a numerical scheme for the hydrodynamics.

Roe's method is one of the representative method for the approximate Riemann solver.

Let's linearize the Euler equation, i.e., freezing the Jacobian Matrix by

$$\partial_t \vec{U} + \partial_x \vec{F} = 0 \ \Rightarrow \ \partial_t \vec{U} + A \partial_x \vec{U} = 0,$$

where $A = \dfrac{\partial F}{\partial U}$ is Jacobian Matrix. Also it is homogeneous, i.e., $\vec{F} = A\vec{U}$

# 7. Roe's method

Roe proposed a matrix $\tilde{A}$ which satisfies the following condition
(i) For $U_L \to U,\ u_R \to U,\ \tilde{A}(U_L, U_R) \to A$
(ii) For an arbitrary $U_{R/L}$, $F(U_R) - F(U_L) = \tilde{A}(U_L, U_R)(U_R - U_L)$
(iii) The eigen vector of $\tilde{A}$ is linearly independent.

(i) and (iii) sound natural requirement. The condition (ii) is related to the Rankine-Hugoniot relation $-\lambda \Delta U + \Delta F = 0$, i.e., the shock condition.

We diagonalize the matrix such that

$\tilde{A} = R\tilde{\Lambda}R^{-1}$ where $\tilde{\Lambda}$ =diag$(\lambda_1, \lambda_2, \lambda_3)$ and $R/R^{-1}$ is the right/left eigen vector.

Therefore, if the Rankine-Hugoniot relation is satisfied, the condition (ii) is satisfied.

# 7. Roe's method

$$\partial_t \vec{U}_i + \frac{1}{\Delta x}\left[\tilde{\vec{F}}_{i+\frac{1}{2}} - \tilde{\vec{F}}_{i-\frac{1}{2}}\right] = 0$$

With this matrix, the numerical flux is calculated by

$$\tilde{\vec{F}}_{i+\frac{1}{2}} = \frac{1}{2}\left[\vec{F}_i + \vec{F}_{i+1} - |\tilde{A}|_{i+\frac{1}{2}}\left(\vec{U}_{i+1} - \vec{U}_i\right)\right],$$

$$\text{where } |\tilde{A}|_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}|\tilde{\Lambda}|_{i+\frac{1}{2}}R^{-1}_{i+\frac{1}{2}}$$

Before going to the detail of the matrix component, let's understand why this scheme works. Suppose that the Jacobian matrix and eigen vectors are frozen,

$$\partial_t \vec{U} + A\partial_x \vec{U} = 0,$$

By multiplying $R^{-1}$,

$$\partial_t\left(R^{-1}\vec{U}\right) + R^{-1}A\partial_x\vec{U} = 0 \Rightarrow \partial_t\left(R^{-1}\vec{U}\right) + \Lambda\partial_x\left(R^{-1}\vec{U}\right) = 0$$

# 7. Roe's method

$$\partial_t \left(R^{-1}\vec{U}\right) + \Lambda\partial_x \left(R^{-1}\vec{U}\right) = 0$$

is the three scalar equation for ($R^{-1}U$). We can apply the method learned in lecture 1. The numerical flux with the 1st-order upwind scheme is

$$\tilde{f}_{i+\frac{1}{2}} = \left(\Lambda R^{-1}\vec{U}\right)_{i+\frac{1}{2}} = \frac{1}{2}\left[\Lambda\left(R^{-1}\vec{U}\right)_i + \Lambda\left(R^{-1}\vec{U}\right)_{i+1} - |\Lambda|R^{-1}\left(\vec{U}_{i+1} - \vec{U}_i\right)\right]$$

c.f.

$$\tilde{f}_{i+\frac{1}{2}} = \frac{1}{2}\left[(f_{i+1} + f_i) - |a|\left(u_{i+1} - u_i\right)\right]$$

If we multiply by R

$$\left(R\Lambda R^{-1}\vec{U}\right)_{i+\frac{1}{2}} = \frac{1}{2}\left[\left(R\Lambda R^{-1}\right)\vec{U}_i + \left(R\Lambda R^{-1}\right)\vec{U}_{i+1} - R|\Lambda|R^{-1}\left(\vec{U}_{i+1} - \vec{U}_i\right)\right]$$

# 7. Roe's method

$$\left(R\Lambda R^{-1}\vec{U}\right)_{i+\frac{1}{2}} = \frac{1}{2}\left[\left(R\Lambda R^{-1}\right)\vec{U}_i + \left(R\Lambda R^{-1}\right)\vec{U}_{i+1} - R|\Lambda|R^{-1}\left(\vec{U}_{i+1} - \vec{U}_i\right)\right]$$

Because $\vec{F} = A\vec{U} = (R\Lambda R^{-1})\vec{U}$

$$\tilde{F}_{i+\frac{1}{2}} = \frac{1}{2}\left[\vec{F}_i + \vec{F}_{i+1} - R|\Lambda|R^{-1}\left(\vec{U}_{i+1} - \vec{U}_i\right)\right]$$

This is nothing but the numerical scheme of the Roe's scheme.

Let's derive the concrete form of the matrix $\tilde{A}$.
First, we define the parameter vector

$$W \equiv \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\rho} \\ \sqrt{\rho}u \\ \sqrt{\rho}H \end{pmatrix}$$ where $H$ is the specific enthalpy.

# 7. Roe's method

With this vector,

$$\vec{U} \equiv \begin{pmatrix} w_1^2 \\ w_1 w_2 \\ \frac{1}{\gamma} w_1 w_3 + \frac{\gamma-1}{2\gamma} w_2^2 \end{pmatrix}, \ \vec{F} = \begin{pmatrix} w_1 w_2 \\ \frac{\gamma-1}{\gamma} w_1 w_3 + \frac{\gamma+1}{2\gamma} w_2^2 \\ w_2 w_3 \end{pmatrix}$$

Let's define

$$\Delta q \equiv q_R - q_L, \ \bar{q} = \frac{q_R + q_L}{2} \text{ for an arbitrary } q_{L/R}.$$

$$\Delta \vec{U} = \vec{U}_R - \vec{U}_L = \begin{pmatrix} \Delta(w_1^2) \\ \Delta(w_1 w_2) \\ \Delta(\frac{1}{\gamma} w_1 w_3 + \frac{\gamma-1}{2\gamma} w_2^2) \end{pmatrix} = \begin{pmatrix} 2\bar{w_1}\Delta(w_1) \\ \bar{w_2}\Delta(w_1) + \bar{w_1}\Delta(w_2) \\ \frac{1}{\gamma}\bar{w_3}\Delta(w_1) + \frac{\gamma-1}{\gamma}\bar{w_2}\Delta w_2 + \frac{1}{\gamma}\bar{w_1}\Delta(w_2) \end{pmatrix}$$

# 7. Roe's method

With the matrix B

$$B = \begin{pmatrix} 2\bar{w}_1 & 0 & 0 \\ \bar{w}_2 & \bar{w}_1 & 0 \\ \frac{1}{\gamma}\bar{w}_3 & \frac{\gamma-1}{\gamma}\bar{w}_2 & \frac{1}{\gamma}\bar{w}_1 \end{pmatrix}$$

$$\Delta\vec{U} = B\Delta\vec{W}$$

Similarly, for F with the matrix C

$$C = \begin{pmatrix} \bar{w}_2 & \bar{w}_1 & 0 \\ \frac{\gamma-1}{\gamma}\bar{w}_3 & \frac{\gamma+1}{\gamma}\bar{w}_2 & \frac{\gamma-1}{\gamma}\bar{w}_1 \\ 0 & \bar{w}_3 & \bar{w}_2 \end{pmatrix}$$

$$\Delta\vec{F} = C\Delta\vec{W},$$

Then, $\Delta\vec{F} = CB^{-1}\Delta\vec{U}$.

Therefore, in the condition (ii), $\tilde{A}(U_R, U_L) = CB^{-1}$

# 7. Roe's method

$$CB^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\frac{\bar{w_2}^2}{\bar{w_1}^2} & (3-\gamma)\frac{\bar{w_2}}{\bar{w_1}} & \gamma-1 \\ (\frac{\gamma-1}{2}\frac{\bar{w_2}^2}{\bar{w_1}^2} - \frac{\bar{w_3}}{\bar{w_1}})\frac{\bar{w_2}}{\bar{w_1}} & \frac{\bar{w_3}}{\bar{w_1}} - (\gamma-1)\frac{\bar{w_2}^2}{\bar{w_1}^2} & \gamma\frac{\bar{w_2}}{\bar{w_1}} \end{pmatrix}$$

and

$$\begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \end{pmatrix} = \begin{pmatrix} (\sqrt{\rho_L} + \sqrt{\rho_R})/2 \\ (\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R)/2 \\ (\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R)/2 \end{pmatrix}$$

If we introduce the Roe average

$$\bar{\rho} \equiv \sqrt{\rho_L \rho_R}$$

$$\bar{u} \equiv \frac{\bar{w}_2}{\bar{w}_1} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\bar{H} \equiv \frac{\bar{w}_3}{\bar{w}_1} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{A} = CB^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\bar{u}^2 & (3-\gamma)\bar{u} & \gamma-1 \\ (\frac{\gamma-1}{2}\bar{u}^2 - \bar{H})\bar{u} & \bar{H} - (\gamma-1)\bar{u}^2 & \gamma\bar{u} \end{pmatrix}$$

# 7. Roe's method

We can easily confirm the condition (i) and (iii) are satisfied

(i) For $U_L \to U,\ u_R \to U,\ \tilde{A}(U_L, U_R) \to A$
(iii) The eigen vector of $\tilde{A}$ is linearly independent.

Extension to higher-order is straightforward as we have learned in Lecture 1. For example, with MUSCL scheme with PPM reconstruction,

$$\tilde{F}_{i+\frac{1}{2}} = \frac{1}{2}\left[F(U_R) + F(U_L) - |\tilde{A}|_{i+\frac{1}{2}}(U_R - U_L)\right],$$

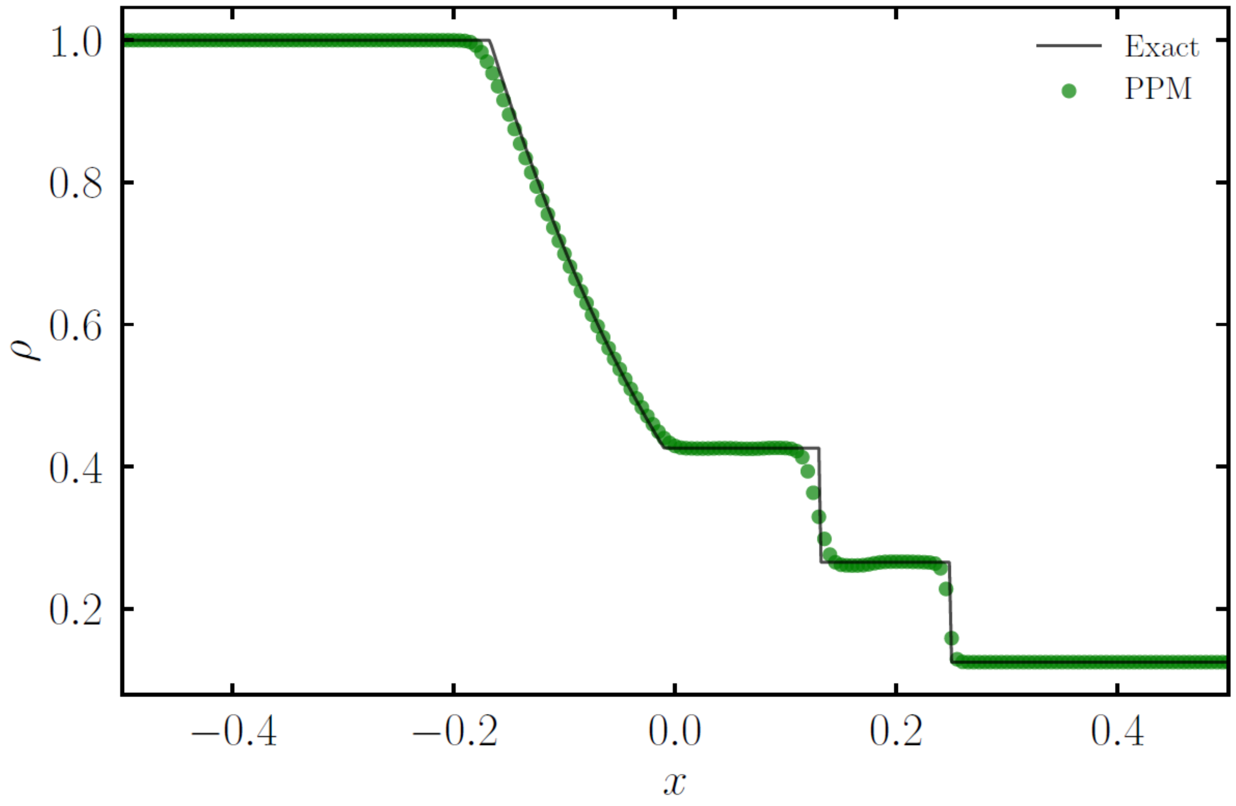
$$(U_L)_{i+\frac{1}{2}} = U_i + \frac{1}{6}\Phi(r_{i-1}^{+})\Delta_{i-1} + \frac{1}{3}\Phi(r_i^{-})\Delta_i,$$

$$(U_R)_{i+\frac{1}{2}} = U_{i+1} - \frac{1}{6}\Phi(r_i^{+})\Delta_i - \frac{1}{3}\Phi(r_{i+1}^{-})\Delta_{i+1},$$

where $\Delta_i = U_{i+1} - U_i,\ r_i^{\pm} = \dfrac{\Delta_{i\pm1}}{\Delta_i}$

# 7. Roe's method

## Example (Sod's problem)



# 8. Summary

We have learned a basic of the numerical hydrodynamics.

1. The 1st-order up-wind scheme is necessary to capture the discontinuity correctly without numerical oscillations.

2. Monotonicity-preserving non-linear scheme is necessary for the higher-order scheme.

3. The Euler equation has three characteristic and the Rankine-Hugoniot relation is essential to capture the shock.

4. By spectral decomposition of the Jacobian Matrix of the Euler equation, we can utilize the technique invented in the scalar equation. Roe's scheme is the representative approximate Riemann solver.

# 8. Summary

Extension to more complicated system will be done (not straightforward) along the same procedure learned here.

Extension 1. Magnetohydrodynamics (7 waves appear)

Extension 2. Special Relativistic Hydrodynamics (Conservative to Primitive conversion + Tabulated EOS)

Extension 3. Special Relativistic Magnetohydrodynamics (7 waves and C to P conversion + Tabulated EOS)

Extension 4. General Relativistic Hydrodynamics (Spacetime curvature)

Extension 5. General Relativistic Magnetohydrodynamics (Spacetime curvature)