PARTIAL DIFFERENTIAL EQUATIONS

TYPES

1) • Elliptic Eq.

e.g. $\nabla^2 u(\vec{x}) = \rho(\vec{x})$ $\vec{x} \in \Omega$  $\Omega \subset \mathbb{R}^3$

space only

one BC on every boundary

Dirichlet $u(\vec{x}) = A(\vec{x}), \quad \vec{x} \in \partial\Omega$

von Neuman $\frac{\partial u}{\partial n}(\vec{x}) = B(\vec{x}) \quad \vec{x} \in \partial\Omega$

asymptotic falloff $u(\vec{x}) \to 0, \quad |\vec{x}| \to \infty$

solve "all at once"

More generally

$$A^{ij}(\vec{x}) \frac{\partial^2 u}{\partial x^i \partial x^j} + F(u, \partial_i u, \vec{x}) = 0,$$

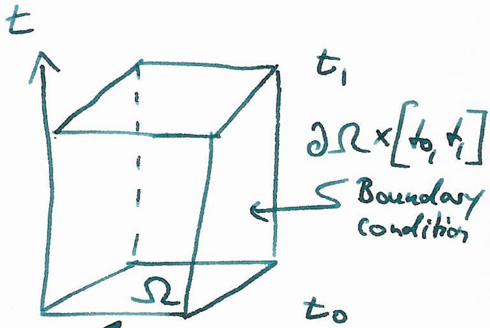
$A^{ij}(\vec{x})$ positive definite

central for constructing initial data for NR

2) Parabolic Eqns

prototype difusion eq

$$\frac{\partial T(\vec{x},t)}{\partial t} - \eta \nabla^2 T(\vec{x},t) = \rho(\vec{x},t)$$



at t_0: one initial condition $T(\vec{x}, t_0)$

infinite propagation speeds

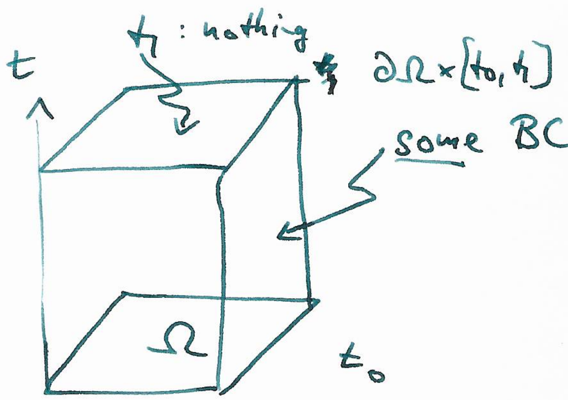
mostly irrelevant for NR (violate causality), except for Apparent Horizon finding

3) Hyperbolic PDE

prototype Wave-eq

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \nabla^2 \psi = S$$

$\psi(\vec{x}, t)$ $\quad \Omega \times \mathbb{R}$ ← time



two initial conditions
$\psi(\vec{x}, t_0)$, $\dot{\psi}(\vec{x}, t_0)$

information propagates with finite velocity

$\Rightarrow$ Evolution of GR, hydro, E+M, ...

- First order reduction

define $\phi_i \equiv c\partial_i \psi$ $\quad \Rightarrow c^2\nabla^2\psi = c^2\delta^{ij}\partial_i\partial_j\psi = c\,\delta^{ij}\partial_i\phi_j$

$\Pi \equiv -\partial_t\psi$ $\quad \Rightarrow \partial_i\Pi = -\partial_i\partial_t\psi = -\partial_t\partial_i\psi = -\frac{1}{c}\partial_t\phi_i$

$$(\partial_t^2 - c^2\nabla^2)\psi = S \iff \begin{cases} \partial_t\psi = -\Pi \\ \partial_t\Pi + c\,\delta^{ij}\partial_i\phi_j = -S \\ \partial_t\phi_i + c\,\partial_i\Pi = 0 \end{cases}$$

general 1st order form: $\underline{u}$ = vector of variables e.g. $\underline{u} = \{\psi, \Pi, \phi_i\}$

$$\partial_t \underline{u} + \underline{\underline{A}}^j \partial_j \underline{u} = \underline{F} \qquad (*)$$

may depend on $\vec{x}$, t, $\underline{u}$. But not $\partial_i \underline{u}$

Examine local properties of solution $\underline{u}(\vec{x},t)$ via plane waves with $k = \frac{2\pi}{\lambda} \to \infty$ (i.e. short wavelengths)

Ansatz

$$\underline{u}(\vec{x},t) = \underline{u}_0 \, e^{i(\vec{k}\vec{x} - \omega t)} = \underline{u}_0 \, e^{ik(\hat{n}\vec{x} - vt)}$$

$\hat{n}$ = direction of propagation

$v = \frac{\omega}{k}$ phase-velocity

in ⊛

$$\left(-ikv\,\underline{u}_0 + ik\,\underline{\underline{A}}^j \hat{n}_j\,\underline{u}_0\right) e^{ik(\hat{n}\vec{x} - vt)} = \underline{F} \qquad \Big| : ik$$

$$-v\,\underline{u}_0 + \underline{\underline{A}}^j \hat{n}_j\,\underline{u}_0 = \underbrace{\frac{1}{ik}\,\underline{F}\,e^{-ik(\dots)}}_{\to 0 \text{ as } k \to \infty}$$

$$\Rightarrow \underbrace{\left(\underline{\underline{A}}^j \hat{n}_j\right)}_{P(n)} \underline{u}_0 = v\,\underline{u}_0$$

Eigenvalue Problem

- if $P(n)$ complete set of e'vectors and all real e'values
  - $\Rightarrow$ full representation of $\underline{u}(\vec{x},t)$ as sum of oscillatory, travelling modes
  - $\Rightarrow$ good "strongly hyperbolic"

- conveniently satisfied if $\underline{\underline{A}}^j \hat{n}_j$ is symmetric or symmetrizable "symmetric hyperbolic"

- for $\hat{n} \perp \partial\Omega$ (outward), modes with $v > 0$ leave ~~down~~ Ω

$\Rightarrow$ BCs only on modes with $v < 0$. "incoming modes"

- if $\underline{\underline{A}}^j \hat{n}_j$ has real e'values, but no complete set of e'vectors
  "weakly hyperbolic"

Thm: symmetric & strongly hyperbolic systems are well-posed.
Weakly hyperbolic ones are <u>not</u>.

well-posed $\cong$ $\exists$ solution

solution is unique

solution depends <u>continuously</u> on initial data

solution is bounded

$$\| u(\cdot, t) \| < e^{Gt} \| u(\cdot, 0) \|$$

Example

$$\dot{u}_1 + u_2' = 0 \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\dot{u}_2 = 0$$

e'values $\lambda_1 = \lambda_2 = 0$

only e'vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ weakly hyperbolic

take $u_1(x,0) = 0$, $u_2(x,0) = \varepsilon e^{ikx}$ $\Rightarrow$ $\| u(\cdot, 0) \| \sim \varepsilon$

but $\dot{u}_1 = -u_2' = -ik\varepsilon e^{ikx}$ $\Rightarrow$ $u_1(x,t) = -ik\varepsilon e^{ikx} t$, $\| u(\cdot, t) \| \sim \varepsilon k t$

arbitrarily fast blow-up

the higher numerical resolution, the smaller k are representable the faster the soln blows up

NUMERICAL SOLUTIONS

Big Q: 1) How to approximate $\begin{cases} u(\vec{x}) \\ u(\vec{x},t) \end{cases}$ by $\begin{cases} u_h(\vec{x}) \\ u_h(\vec{x},t) \end{cases}$

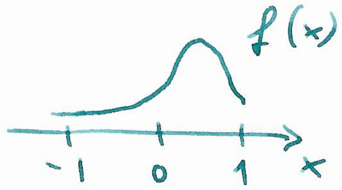
with finite # of DoF ?

2) In what sense does $\begin{cases} u_h(\vec{x}) \\ u_h(\vec{x},t) \end{cases}$ satisfy the original PDE ?

first example : 1-D Laplace Eqn

$$u''(x) = \rho(x) \qquad x \in [-1,1]$$

s.t. solution is $f(x) = e^{-\frac{(x-0.3)^2}{w^2}}$.



i.e. $\rho(x) = f''(x) = \left(-\frac{2}{w^2} + \frac{4(x-0.3)^2}{w^4}\right) f(x)$

BCs $\quad u(-1) = f(-1)$

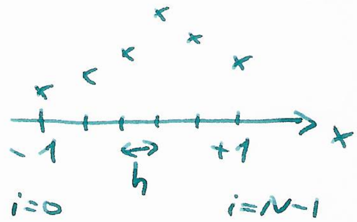
$\quad\quad u(1) = f(1)$

Finite-Differences

N Equally spaced points $x_i = -1 + h\cdot i \qquad h = \frac{2}{N-1}, \quad i = 0, \dots, N-1$

represent solution at x_i

$$u(x) \longrightarrow u(x_i) =: u_i \qquad i = 0, \dots, N-1$$



approximate derivs by finite differences

$$\partial_x u(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2)$$

$$\partial_x^2 u(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

$$u'' = g \implies \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = g(x_i) \qquad i = 1, 2, \dots, N-2$$

BC $\quad u_0 = f(-1)$

$\quad\quad u_{N-1} = f(+1)$

$$\frac{1}{h^2}\begin{pmatrix} h^2 & & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & & h^2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f(-1) \\ g_1 \\ g_2 \\ \vdots \\ g_{N-2} \\ f(+1) \end{pmatrix}$$

Matrix Eq $\qquad A \cdot u = b \implies$ Solve (~~[illegible]~~ see Tim)

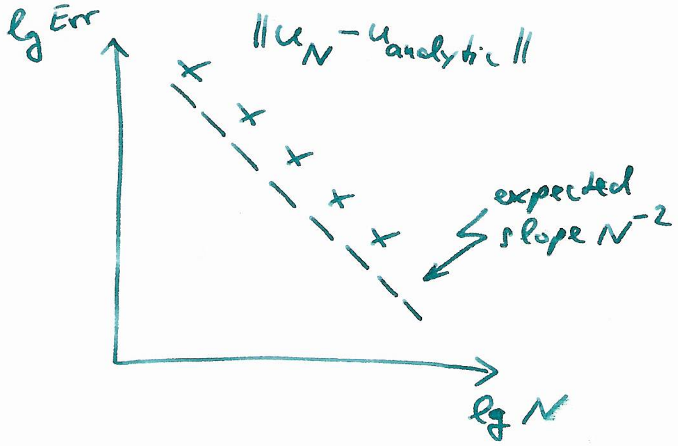
= PDE_1D_Poisson.ipynb (part 1) =

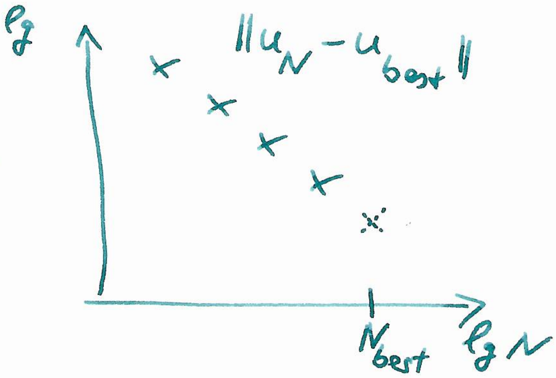
IMPORTANT: CHECK CONVERGENCE

COMPARE W/ EXPECTATIONS

→ strong test of correctness

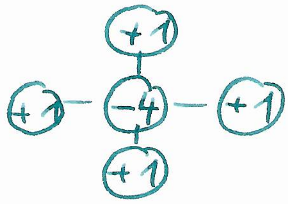
→ tells you how good soln is





2nd order FD:

$$u'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \Rightarrow Err \propto h^2 \propto N^{-2}$$



Laplace Eq in 2-D with FD

- very similar $\nabla^2 u_{i,j} = \dfrac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$
- length (b) = N^2 # of grid-points
- ~~size~~ shape (A) = N^2, N^2 total N^4 entries !
- indexing more difficult

= PDE_2D_Poisson.ipynb =

Curse of dimensionality

N - pts per dim

N^D - # of grid-points

N^{2D} - # of elements in A

$O(N^{3D})$ - cost of LU decomp of A.

N^6 in 2D

N^9 in 3D ☹

SPECTRAL METHODS

approx $u(x,t) \Rightarrow u^{(N)} = \sum_{k=0}^{N-1} \tilde{u}_k(t)\phi_k(x)$

Fourier
Chebyshev
$Y_{\ell m}$

smooth solutions $\Rightarrow$ exponential convergence

$\| u^{(N)} - u \| \to e^{-cN}$ $\qquad \tilde{u}_k \sim e^{-ck}$

exact derivatives

$$\partial_x u^{(N)}(x) = \sum_k \tilde{u}_k \underbrace{\partial_x \phi_k(x)}_{\tilde{D}_{\ell k}\phi_\ell(x)} = \sum \tilde{u}'_\ell \phi_\ell$$

$\tilde{D}_{\ell k}\tilde{u}_k$

physical $\leftrightarrow$ spectral

$x_i \quad i = 0, \dots N-1 \quad$ non-uniform

$$u_i := u(x_i) = \sum \tilde{u}_k \underbrace{\phi_k(x_i)}_{\equiv A_{ik}} = A_{ik}\tilde{u}_k$$

also $\tilde{u}_k = (A^{-1})_{ki} u_i$

good choice $\{x_i\} \Leftrightarrow$ well-conditioned A_{ik}
& trafo doable by FFT

Boyd: "Chebyshev and Fourier Spectral Methods"

Demo: Laplace eq in 1D with Chebyshev

$$T_k(x) = \cos(k \arccos x) \qquad x \in [-1, 1]$$

$$x_i = \cos \frac{\pi i}{N-1} \qquad i = 0, \dots, N-1$$

clustered at ± 1

$$A_{ik} = T_k(x_i)$$

$$A^{-1}_{kj} = \frac{2}{(N-1)\bar{c}_k \bar{c}_j} T_k(x_j) \qquad \bar{c}_k = \begin{cases} 2 & k = 0, N-1 \\ 1 & \text{else} \end{cases}$$

following Kidder +, PRD 62 p. 084026 (2000)

Discrete orthogonality

$$\sum_i \frac{2}{(N-1)\bar{c}_k \bar{c}_i} T_k(x_i) T_j(x_i) = \delta_{kj}$$

Differentiation by recurrence formulae

$$u = \sum \tilde{u}_k T_k \quad , \quad \tilde{u}' = \sum \tilde{u}'_k T_k$$

$$\tilde{u}'_k = \frac{1}{c_k}\left(\tilde{u}'_{k+2} + 2(k+1)\tilde{u}_{k+1}\right) \qquad c_k = \begin{cases} 2 & k=0 \\ 1 & \text{else} \end{cases}$$

seed with $\tilde{u}'_N = 0 \quad \tilde{u}'_{N-1} = 0$

deriv of polynomial reduces order

$$\Rightarrow \tilde{u}'_k = \tilde{D}_{k\ell} \tilde{u}_\ell$$

Deriv at grid-points

$$u'(x_i) =: u'_i = A_{ik} \tilde{D}_{k\ell} A^{-1}_{\ell j} u_j$$

$$u'' = \rho \iff (A \tilde{D}^2 A^{-1})_{ji} u_i = \underbrace{\rho(x_j)}_{b_j :=}$$

modify by BC $\Rightarrow$ Matrix Eq again

≡ PDE_1D_Poisson.ipynb Part 2 ≡