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Course webpage: https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/

Homework due date: Homeworks are due on Friday 26/02/2021 at the following address: https://moodle.hu-berlin.de/mod/assign/view.php?id=2692284

Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

I. FROM SCHWARZSCHILD GEODESICS TO THE 2PM ARBITRARY-MASS SCATTERING ANGLE

We have argued that the result for the arbitrary-mass scattering angle through $\mathcal{O}(G^2)$ must take the form ($c = 1$ again)

$$\frac{M}{E}\chi = \frac{GM}{b}X_1(\gamma) + \Big(\frac{GM}{b}\Big)^2 X_2(\gamma) + \mathcal{O}(G^3), \tag{1}$$

where again $M = m_1 + m_2$, and where the functions X_k depend *only* on the Lorentz factor γ (not also on the mass ratio). It is thus determined by its test-body limit, the scattering angle function for geodesics in a Schwarzschild background with mass m,

$$\chi_{\rm Schw} = \frac{Gm}{b}X_1(\gamma) + \Big(\frac{Gm}{b}\Big)^2 X_2(\gamma) + \mathcal{O}(G^3), \tag{2}$$

where γ is the conserved energy per mass and $b = \ell/\sqrt{\gamma^2 - 1}$ where ℓ is the conserved angular momentum per mass.

A handy canonical Hamiltonian for Schwarzschild geodesics, for a test mass of (irrelevant) mass μ, can found by writing out

$$-\mu^2 = g^{\mu\nu}(x)p_\mu p_\nu, \tag{3}$$

in terms of coordinates $x^\mu = (t, r, \phi)$ and components $p_\mu = (p_t, p_r, p_\phi)$ and solving for p_t; the canonical Hamiltonian is

$$H(r, p_r, p_\phi) = -p_t. \tag{4}$$

(We are using symmetry to specialize to the equatorial plane $\theta = \pi/2$.) The resultant equations of motion are

$$\dot{r} = \frac{\partial H}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r},$$
$$\dot{\phi} = \frac{\partial H}{\partial L}, \qquad \dot{L} = -\frac{\partial H}{\partial \phi} = 0. \tag{5}$$

Again, p_ϕ is a constant of motion, the conserved angular momentum, as $H = -p_t$ is the conserved energy (due to the axial-rotation and time-translation Killing symmetries). We define the mass-rescaled constants

$$\gamma = \frac{-p_t}{\mu}, \qquad \ell = \frac{p_\phi}{\mu}. \tag{6}$$

(a) Explain why the equations of motion resulting from this Hamiltonian are equivalent to the t-parametrized geodesic equation.

(b) Starting from the standard form of the Schwarzschild metric in Schwarzschild coordinates $(t, r_{\rm S}, \phi)$,

$$\mathrm{d}s^2 = -\Big(1 - \frac{2Gm}{r_{\rm S}}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2Gm}{r_{\rm S}}\Big)^{-1}\mathrm{d}r_{\rm S}^2 + r_{\rm S}^2\mathrm{d}\phi^2, \tag{7}$$

transform to isotropic coordinates (t, r, ϕ) via

$$r_{\rm S} = r\Big(1 + \frac{Gm}{2r}\Big)^2, \tag{8}$$

and find the new expression of the metric.

(c) In isotropic coordinates, solve for $\dfrac{p_r^2}{\mu^2}$ as a function of (r, γ, ℓ). [One could first solve for the Hamiltonian function and then solve for p_r^2, but both are defined by "the mass shell constraint" (3).] Expand this expression to quadratic order in G, but then treat that expression as exact (under the coming square root, to make the integral elementary; this is also why we transformed to isotropic coordinates). Apply

$$\pi + \chi(E, L) = -2\int_{r_{\min}}^{\infty} \mathrm{d}r\, \frac{\partial}{\partial L} p_r(r, E, L), \tag{9}$$

do the integral, then re-expand in G, to find the Schwarzshild scattering angle function to 2PM order (2).

(d) Justify the relation $\ell = \gamma v b$ in this context, recalling $\gamma v = \sqrt{\gamma^2 - 1}$.

(e) Deduce the arbitrary-mass scattering angle to 2PM order.

II. NEWTONIAN QUADRUPOLAR TIDAL IMPRINT IN THE GW PHASING

Consider a neutron star-black hole binary system of total mass M and reduced mass μ whose orbital motion is described by Newtonian gravity. The Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} - \frac{1}{2}Q_{ij}\mathcal{E}_{ij} + L_{\rm int}, \tag{10}$$

where $L_{\rm int}$ describes the internal dynamics of the quadrupole and the Newtonian tidal field is

$$\mathcal{E}_{ij} = -m_{\rm BH}\partial_i\partial_j(1/r) = -m_{\rm BH}(3n^i n^j - \delta^{ij})/r^3, \tag{11}$$

where $n^i = x^i/r$ is a unit vector. Note that $n^i n_i = 1$ and $\delta^{ij}\delta_{ij} = 3$. Assume that the quadrupole is adiabatically induced and given by

$$Q_{ij}^{\rm ad} = -\lambda \mathcal{E}_{ij}, \tag{12}$$

where λ is the tidal deformability parameter. The internal Lagrangian then describes only the elastic potential energy $L_{\rm int}^{\rm ad} = -Q_{ij}Q^{ij}/(4\lambda)$. Throughout this exercise, assume that tidal effects are small and can be treated as linear perturbations.

(a) Obtain the equations of motion for r and ϕ from the Euler-Lagrange equations.

(b) Assume that the orbit is circular ($\ddot{r} = 0$ and $\dot{\phi} = \Omega$). Starting from the radial equation of motion, express the radius as $r(\Omega) = M^{1/3}\Omega^{-2/3}(1 + \delta r)$ and compute the linear tidal corrections δr.

1. Calculate the energy of the system from (10). Specialize to adiabatic quadrupoles and circular orbits, and express the energy in terms of Ω.

(c) The leading order gravitational radiation is generated by the total quadrupole of the system $Q_{ij}^T = Q_{ij}^{\rm orbit} + Q_{ij}$. Compute the tidal contribution to the energy flux from the quadrupole formula.

(d) In the stationary phase approximation (SPA) for the gravitational wave signal, the phasing can be computed from the formula

$$\frac{d^2\Psi_{\rm SPA}}{d\Omega^2} = 2\frac{dE/d\Omega}{\dot{E}_{\rm GW}}. \tag{13}$$

Compute the tidal contribution to $\Psi_{\rm SPA}$, to linear order in the tidal effects. Express your result in terms of the post-Newtonian parameter $x = (M\Omega)^{2/3} = (\pi M f_{\rm GW})^{2/3}$ and show that the tidal phase correction scales as x^5 relative to the leading order phasing.