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Course webpage: https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/

Homework due date: Homeworks must be uploaded before Monday 08/02/2021 at the following address: https://moodle.hu-berlin.de/mod/assign/view.php?id=2673402

Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

Recommended readings:

1. A. Buonanno and T. Damour, Phys. Rev. **D59** (1999) 084006.
2. A. Buonanno and T. Damour, Phys.Rev. **D62** (2000) 064015.
3. B. F. Schutz and C. M. Will, Astrophys.J.Lett. **291** (1985), L33-L36
4. V. Ferrari and B.Mashhoon, Phys. Rev. **D30** (1984), 295

I. BLACK-HOLE QUASI-NORMAL MODES

In the lectures and the previous tutorial session, it was shown that the quasinormal modes (QNMs) of a Schwarzschild black hole are characterized by complex frequencies $\omega = \omega_R + i\omega_I$, with ω_R and ω_I the real and the imaginary parts, respectively.

(a) Use Table I from arXiv:gr-qc/0411025 to plot ω_R and ω_I of the quadrupolar mode ($l = 2$) versus n, where n is the overtone number that identifies the number of nodes in the radial wavefunction (plus 1 in the reference's conventions). Use n = 1–12, 20, 30, 40. [Note that the values in Table I correspond to $(\omega_R, -\omega_I)$ in our conventions, given the time-dependence of the QNMs as $e^{i\omega t}$.]

Your plot should exhibit some features which could be considered strange according to certain intuition, interpreting ω_R as an oscillation frequency and ω_I as a decay rate. For typical systems with a set of vibrational modes, like a string or an elastic body, both the oscillation frequency and the decay rate increase with increasing overtone number, i.e. with an increasing number of nodes in the wavefunction. The QNM plot, however, shows that ω_R is first decreasing with n, then has a zero, and then increases to an asymptotically constant value. This behavior can be seen as somewhat less mysterious by reinterpreting ω_R and ω_I as follows.

(b) Consider a simple damped oscillator with amplitude $\psi(t)$ obeying

$$\ddot{\psi} + \gamma_0 \dot{\psi} + \omega_0^2 \psi = 0. \tag{1}$$

Writing the two linearly independent solutions as $\exp((\pm i\omega_R - \omega_I)t)$, find the relationship between ω_R, ω_I and ω_0, γ_0. Invert this relation, make plots of ω_0 and γ_0 versus n for the Schwarzschild QNMs and comment how this interpretation alleviates the above discussion.

II. ON THE EFFECTIVE-ONE-BODY HAMILTONIAN AND DYNAMICS

We have derived in class the mapping between the *real* PN Hamiltonian and the *effective* Hamiltonian using the Hamilton-Jacobi formalism. Here we want to construct the effective-one-body (EOB) Hamiltonian using a canonical transformation.

Using reduced (or dimensionless) variables $\mathbf{Q}, \mathbf{P}$ and $\hat{H}_{\rm eff}$, the EOB Hamiltonian reads

$$\hat{H}_{\rm eff}(Q, P) = c^2 \sqrt{A(Q) \left[1 + \frac{1}{c^2}\,\mathbf{P}^2 + \left(\frac{A(Q)}{D(Q)} - 1\right) \frac{1}{c^2}\,(\mathbf{N}\cdot\mathbf{P})^2\right]}\,, \tag{2}$$

where $\mathbf{N} = \mathbf{Q}/Q$ and

$$A(Q) = 1 + \frac{a_1}{c^2\,Q} + \frac{a_2}{c^4\,Q^2} + \frac{a_3}{c^6\,Q^3} + \cdots\,, \tag{3}$$

$$D(Q) = 1 + \frac{d_1}{c^2\,Q} + \frac{d_2}{c^4\,Q^2} + \cdots\,, \tag{4}$$

where a_i, d_i are unknown coefficients that will be determined by the mapping to the (reduced) PN Hamiltonian

$$\hat{H}_{\rm real}(q, p) = \hat{H}_{\rm Newt}(q, p) + \frac{1}{c^2}\,\hat{H}_{\rm 1PN}(q, p) + \cdots\,, \tag{5}$$

$$\hat{H}_{\rm Newt}(q, p) = \frac{1}{2}\mathbf{p}^2 - \frac{1}{q}\,, \tag{6}$$

$$\hat{H}_{\rm 1PN}(q, p) = -\frac{1}{8}(1 - 3\nu)\,\mathbf{p}^4 - \frac{1}{2q}\left[(3 + \nu)\,\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\right] + \frac{1}{2q^2}\,, \tag{7}$$

where $\mathbf{q}$ and $\mathbf{p}$ are reduced variables, $\mathbf{n} = \mathbf{q}/q$ and $\nu = m_1\,m_2/(m_1 + m_2)^2$, being m_1 and m_2 the black-hole masses. At 1PN order the real and effective Hamiltonians are related as

$$1 + \frac{\hat{H}_{\rm real}(q, p)}{c^2}\left(1 + \alpha_1\,\frac{\hat{H}_{\rm real}(q, p)}{c^2}\right) = \frac{\hat{H}_{\rm eff}(Q(q, p), P(q, p))}{c^2}\,, \tag{8}$$

where α_1 is an unknown coefficient that will be determined by the mapping. The canonical transformation at 1PN order is

$$Q^i = q^i + \frac{1}{c^2}\,\frac{\partial G_{\rm 1PN}}{\partial p_i}\,, \tag{9}$$

$$P_i = p_i - \frac{1}{c^2}\,\frac{\partial G_{\rm 1PN}}{\partial q^i}\,, \tag{10}$$

with

$$G_{1\mathrm{PN}}(\mathbf{q},\mathbf{p}) = (\mathbf{q}\cdot\mathbf{p})\,\left[c_1\mathbf{p}^2 + \frac{c_2}{q}\right]\,, \tag{11}$$

where c_1, c_2 are unknown coefficients that will be determined by the mapping.

The goal of this exercise is to determine α_1, c_1, c_2 as a function of ν.
Insert the canonical transformation given in Eqs. (9) and (10) in Eq. (8) and expand the latter in PN orders through 1PN order. By equating terms with the same structures in $\mathbf{q}$, $\mathbf{p}$, derive the equations for the unknown coefficients α_1, c_1, c_2 and set $a_2 = a_3 = ... = a_n = d_1 = d_2 = ... = d_n = 0$. In this case you should find that: $\alpha_1 = \nu/2$, $c_1 = -\nu/2$ and $c_2 = 1 + \nu/2$. [Hint: introduce the parameter $\epsilon^2 \equiv 1/c^2$, work with the square of Eq. (8) to get rid of the square root in Eq. (2), and neglect the terms with order higher than $O(\epsilon^4)$. Note that it is sufficient to derive $Q \equiv |\mathbf{Q}| = \sqrt{Q^i\,Q_i}$, $P \equiv |\mathbf{P}| = \sqrt{P^i\,P_i}$ and $\mathbf{N}\cdot\mathbf{P} = N^i\,P_i$ as function of $q \equiv |\mathbf{q}|$, $p \equiv |\mathbf{p}|$ and $\mathbf{n}\cdot\mathbf{p}$ through 1PN order using the canonical transformation given in Eqs. (9) and (10).]