Gravitational-Wave Course — Homework Sheet 5

Instructor: Alessandra Buonanno (alessandra.buonanno@hu-berlin.de)
Guest Lecturer: Jan Steinhoff (jan.steinhoff@hu-berlin.de)
Guest Lecturer: Justin Vines (justin.vines@hu-berlin.de)
Tutor : Lorenzo Speri (lorenzo.speri@hu-berlin.de)
Tutor *(corresponding for this sheet)* **:** Stefano Savastano (stefano.savastano@aei.mpg.de)

Course webpage: https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/

Homework due date: Homeworks must be uploaded before Monday 25/01/2021 at the following address: https://moodle.hu-berlin.de/mod/assign/view.php?id=2647491

Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

Recommended readings:

1. Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory: T. Damour, 1609.00354
2. Classical and quantum scattering in post-Minkowskian gravity: T. Damour, 1912.02139
3. Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order: J. Vines, J. Steinhoff, A. Buonanno, 1812.00956
4. Black Hole Binary Dynamics from the Double Copy and Effective Theory: Z. Bern *et al.*, 1908.01493
5. From Boundary Data to Bound States II: Scattering Angle to Dynamical Invariants (with Twist): G. Kälin, R. Porto, 1911.09130

I. SCATTERING OF TWO POINT MASSES IN THE FIRST POST-MINKOWSKIAN (1PM) APPROXIMATION

We have argued that an effective action for two point masses coupled to gravity is given by

$$\mathcal{S} = \int d^4x\sqrt{-g}\frac{R}{16\pi G} - \sum_{a=1,2} m_a \int d\tau_a \sqrt{-g_{\mu\nu}(x_a)\dot{x}_a^\mu \dot{x}_a^\nu}, \tag{1}$$

with $c = 1$. The degrees of freedom are the spacetime metric $g_{\mu\nu}(x)$ and the masses' (arbitrarily parametrized) worldlines $x = x_a(\tau_a)$ with tangents $\dot{x}_a^\mu = dx_a^\mu/d\tau_a$; the masses m_a are constants. Varying with respect to (wrt) the metric yields Einstein's field equation,

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu}, \tag{2}$$

with the effective energy-momentum tensor for the point masses given by

$$T^{\mu\nu}(x) = \sum_a m_a \int d\tau_a \frac{\dot{x}_a^\mu \dot{x}_a^\nu}{\sqrt{-\dot{x}^2}} \frac{\delta^4(x - x_a)}{\sqrt{-g}}. \tag{3}$$

Varying wrt the worldlines yields effective geodesic equations,

$$\dot{x}_a^\nu \nabla_\nu \Big(\frac{\dot{x}_{a\mu}}{\sqrt{-\dot{x}_a^2}} \Big) = 0, \tag{4}$$

where ∇_μ is the covariant derivative for the (appropriately regularized/renormalized) full dynamical spacetime metric.

We can make a post-Minkowskian ansatz for the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(G^2), \tag{5}$$

where η is a flat Minkowski metric, and the perturbation h is $\mathcal{O}(G)$. It is advantageous to maintain manifest (Lorentz) covariance wrt the background Minkowski spacetime. Henceforth, after defining the metric perturbation above (with down indices), all index manipulations (raising, lowering, vector squares and dot products) will be done with η (unless otherwise stated); e.g., $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. If we also impose the harmonic/Lorenz/de Donder gauge condition,

$$\mathcal{P}^{\mu\nu}{}_{\rho\sigma}\partial_\nu h^{\rho\sigma} = 0, \qquad \mathcal{P}^{\mu\nu}{}_{\rho\sigma} = \delta^{(\mu}{}_\rho \delta^{\nu)}{}_\sigma - \frac{1}{2}\eta^{\mu\nu}\eta_{\rho\sigma}, \tag{6}$$

where $\mathcal{P}$ is the (Minkowski) trace-reversal operator ($\bar{h}^{\mu\nu} = \mathcal{P}^{\mu\nu}{}_{\rho\sigma}h^{\rho\sigma}$), then the field equation becomes

$$\Box h^{\mu\nu} = -16\pi G \mathcal{P}^{\mu\nu}{}_{\rho\sigma}T^{\rho\sigma} + \mathcal{O}(G^2), \tag{7}$$

where $\Box = \partial_\rho\partial^\rho$, with ∂ being the (flat, commuting) covariant derivative for η, and the source becomes

$$T^{\mu\nu} = \sum_a m_a \int d\tau_a \, \dot{x}_a^\mu \dot{x}_a^\nu \, \delta^4(x - x_a) + \mathcal{O}(G), \tag{8}$$

where we now specialize the parameters τ_a to be proper times wrt the Minkowski metric, $\eta_{\mu\nu}\dot{x}_a^\mu\dot{x}_a^\nu = -1$. The effective geodesic equations become

$$\frac{\mathrm{d}p_{a\mu}}{\mathrm{d}\tau_a} = \frac{m_a}{2}\dot{x}^\nu \dot{x}^\rho \partial_\mu h_{\nu\rho}(x_a) + \mathcal{O}(G^2), \qquad p_{a\mu} \equiv m_a g_{\mu\nu}\dot{x}_a^\nu = m_a \dot{x}_{a\mu} + \mathcal{O}(G), \tag{9}$$

where we define the momentum $p_{a\mu}$ by lowering the index on the tangent $\dot{x}_a^\mu$ with the *full* metric (while in the last expression $\dot{x}_{a\mu} = \eta_{\mu\nu}\dot{x}_a^\nu$ according to our conventions of using η, and $p_a^\mu = \eta^{\mu\nu}p_{a\nu}$ below; note that the difference between $\dot{x}_a$ and p_a/m_a would not matter on the right-hand side of the equation of motion).

(a) Verify that (3) becomes (8) and that (4) becomes (9) under the stated assumptions.

The field equation (7) together with the gauge condition (6) imply that the energy-momentum tensor must be (approximately) conserved wrt the background flat metric, $\partial_\mu T^{\mu\nu} = \mathcal{O}(G)$, and this will be true only if the worldlines are unaccelerated ($+\mathcal{O}(G)$). This is indeed implied by the worldlines' equations of motion (9), $\mathrm{d}\dot{x}_a^\mu/\mathrm{d}\tau_a = \mathcal{O}(G)$. The worldlines must then take the form

$$\begin{aligned} x_a^\mu(\tau_a) &= y_a^\mu + u_a^\mu \tau_a + \mathcal{O}(G) \\ &\equiv x_{a0}^\mu(\tau_a) + \mathcal{O}(G), \end{aligned} \tag{10}$$

where y_a^μ and u_a^μ are constant vectors; y_a is the displacement from the origin at $\tau_a = 0$, and u_a is the zeroth-order 4-velocity, with $-1 = u_a^2 = \eta_{\mu\nu} u_a^\mu u_a^\nu$. We can identify $p_{a0}^\mu = m_a u_a^\mu$ as the momentum of the incoming state, at $\tau_a \to -\infty$.

Plugging (10) for mass 2 into (8), its contribution to $T^{\mu\nu}$ is

$$T_2^{\mu\nu} = m_2 u_2^\mu u_2^\nu \int d\tau_2\, \delta^4(x - x_{20}) + \mathcal{O}(G). \tag{11}$$

(b) Using the retarded Green's function for the flat wave operator,

$$\mathcal{G}_{\mathrm{ret}}(x, x') = \delta\Big(\frac{(x-x')^2}{2}\Big) \times \begin{cases} 1, & x \text{ in the future of } x' \\ 0, & \text{otherwise} \end{cases}, \tag{12}$$

satisfying

$$\Box \mathcal{G}_{\mathrm{ret}}(x, x') = -4\pi \delta^4(x - x'), \tag{13}$$

show that the solution to the field equation (7) for the field sourced by mass 2 is

$$h_2^{\mu\nu}(x) = \mathcal{P}^{\mu\nu}{}_{\rho\sigma} u_2^\rho u_2^\sigma \frac{4Gm_2}{r_2(x)} + \mathcal{O}(G^2), \tag{14}$$

where

$$r_2(x) = \sqrt{(x - y_2)^2 + [u_2 \cdot (x - y_2)]^2} \tag{15}$$

is the distance of the field point x from mass 2's zeroth-order worldline, in its rest frame.

For the zeroth-order worldlines, $x_{10}^\mu = y_1^\mu + u_1^\mu \tau_1$ and $x_{20}^\mu = y_2^\mu + u_2^\mu \tau_2$, we can choose the parametrizations such that

$$y_1^\mu - y_2^\mu = b^\mu, \qquad b \cdot u_1 = b \cdot u_2 = 0, \tag{16}$$

which defines b^μ as the vectorial impact parameter separating the zeroth-order worldlines at the points of mutual closest approach (at $\tau_1 = \tau_2 = 0$). The relative Lorentz factor γ between the masses' incoming rest frames is defined by

$$-u_1 \cdot u_2 = \gamma = \frac{1}{\sqrt{1 - v^2}}, \tag{17}$$

where v is the relative velocity between the incoming rest frames.

Now we can solve the equation of motion (9) for mass 1 as perturbed by the field of mass 2, taking $h \to h_2$ in (9) (dropping the infinite self-field contribution from h_1). Everywhere on the right-hand side of (9), we can take $x_1 \to x_{10}$, using the unaccelerated zeroth-order worldline, since the corrections would finally end up at $\mathcal{O}(G^2)$. We find the "impulse" (total change in momentum) to be

$$\Delta p_{1\mu} = \frac{m_1}{2} u_1^\nu u_1^\rho \int_{-\infty}^{\infty} \mathrm{d}\tau_1\, \partial_\mu h_{2\nu\rho}\Big|_{x = x_{10}(\tau_1)} + \mathcal{O}(G^2). \tag{18}$$

(c) Simplify the integrand and perform the integral, showing that the impulse is of the form

$$\Delta p_1^\mu = G m_1 m_2 \frac{b^\mu}{b^2} f(\gamma) + \mathcal{O}(G^2), \tag{19}$$

and find the function $f(\gamma)$. What is the impulse Δp_2^μ for mass 2?

The total energy E and the velocity $u_{\rm cm}$ of the center-of-mass frame are defined by

$$E^2 = -(p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma, \qquad u_{\rm cm}^\mu = \frac{p_1^\mu + p_2^\mu}{E}. \tag{20}$$

Here we use $p_a^\mu = m_a u_a^\mu$ for the asymptotic incoming momenta, and we reason as though we were in flat spacetime (at infinity). The individual momenta can be split into components along and orthogonal to $u_{\rm cm}$ according to

$$\begin{aligned} p_1^\mu &= E_1 u_{\rm cm}^\mu + p^\mu, \qquad\qquad p \cdot u_{\rm cm} = 0, \\ p_2^\mu &= E_2 u_{\rm cm}^\mu - p^\mu. \end{aligned} \tag{21}$$

(d) Find an expression for the magnitude $|p|$ of the spacelike vector p^μ, giving the equal and opposite momenta in the cm frame, in terms of m_1, m_2 and γ.

Note that, if we were to define $\gamma = \dfrac{-p_1 \cdot p_2}{m_1 m_2}$, E, E_a, $|p|$ in terms of the asymptotic outgoing momenta, $p'_a = p_a + \Delta p_a$, they would all be the same as for the incoming state. (Why?)

(e) Argue that the cm-frame scattering angle is given by

$$\chi = \frac{|\Delta p_1|}{|p|} + \mathcal{O}(G^2), \tag{22}$$

and find the expression for χ in terms of m_a, γ and b.