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Course webpage: https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/

Homework due date: Homeworks must be uploaded before Monday 11/01/2021 at the following address: https://moodle.hu-berlin.de/mod/assign/view.php?id=2622752

Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

Recommended readings:

1. Coarse graining and effective field theory (EFT): K. Huang, *Quantum Field Theory—From Operators to Path Integrals*, John Wiley & Sons (New York, 1998); in particular chapter 16.
2. Neutron star physics: http://adsabs.harvard.edu/abs/2004Sci...304..536L
3. post-Newtonian approximation and EFT: M. Levi, Rept. Prog. Phys. **83**, 075901 (2020); arxiv:1807.01699.
4. Fokker action: T. Damour, G. Esposito-Farese, Phys. Rev. D **53** 5541–5578 (1996); arXiv:gr-qc/9506063.

Notice

You are required to work on **two of the three exercises** of your choice for this homework sheet.

I. COARSE-GRAINING AND POINT-PARTICLES

Compact bodies like neutron stars and black holes can be represented by point-particles in general relativity at scales that are much larger than their size. This coarse-graining can be understood as "integrating out" body-scale degrees of freedom (short-wavelength Fourier modes), but this is difficult to work out in detail due to the nonlinear nature of the theory. Let us here gain intuition on this point-of-view in Newtonian theory. Consider a body of size R centered around the position $\mathbf{x}_0$, which is described by a mass density $\rho(\mathbf{x})$ or its Fourier transform $\tilde{\rho}(\mathbf{k}) = \int d^3x\, \rho(\mathbf{x})\, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0)}$. The coarse-grained (spatially averaged) density reads

$$\langle\rho\rangle = \int \frac{d^3k}{(2\pi)^3}\, \tilde{f}(\mathbf{k})\, \tilde{\rho}(\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0)}, \tag{1}$$

where $\tilde{f}(\mathbf{k})$ is a low-pass filter. We take it to be $\tilde{f}(\mathbf{k}) = \Theta(k_{\rm cut}/|\mathbf{k}| - 1)$ for definiteness, where Θ is the Heaviside step function and $k_{\rm cut} \lessapprox 1/R$. For simplicity, we do not denote explicitly a possible time dependence of all quantities.

a) Perform a rescaling by a constant factor C as

$$\mathbf{k} \to C\mathbf{k}, \qquad \mathbf{x} \to \frac{\mathbf{x}}{C}, \qquad \mathbf{x}_0 \to \frac{\mathbf{x}_0}{C}, \qquad \rho \to C^3\rho, \tag{2}$$

keeping R or $k_{\rm cut}$ fixed. The idea is to change the characteristic scale of the system, or the unit of measurement, from R to a larger length scale, say, the orbital separation r of a binary. That is, powers in $C = \frac{R}{r} \ll 1$ correspond to the size of multipole corrections observed at a distance r away from the body. Work out the multipole expansion of $\langle\rho\rangle$ in $C = \frac{R}{r}$ explicitly.

b) What is the significance of the low-pass filter $\tilde{f}(\mathbf{k})$ in this calculation? **[optional!]**

c) The interaction of the body with an external Newtonian gravitational potential $\phi(\mathbf{x})$ is given by a term in the action reading

$$S_{\rm int} = -\int dt d^3x\, \langle\rho\rangle\phi. \tag{3}$$

Using the expansion of $\langle\rho\rangle$ in C, show that this can be written as an action along a worldline of the form

$$S_{\rm int} = -\int dt \left[m(\phi)_{\mathbf{x}=\mathbf{x}_0} + C m_i(\partial_i\phi)_{\mathbf{x}=\mathbf{x}_0} + \frac{C^2}{2} m_{ij}(\partial_i\partial_j\phi)_{\mathbf{x}=\mathbf{x}_0} + \frac{C^3}{3!} m_{ijk}(\partial_i\partial_j\partial_k\phi)_{\mathbf{x}=\mathbf{x}_0} + \mathcal{O}\left(\frac{R^4}{r^4}\right)\right]. \tag{4}$$

Convince yourself that one can remove the term linear in C by a shift of $\mathbf{x}_0$. Remark: The coefficients $m_{ij\dots}$ can be made tracefree using the vacuum field equations $\partial_k\partial_k\phi = 0$. The multipole corrections in the action here are important for modeling tidal effects.

d) We can recover eq. (4) more directly: Start from $S_{\rm int} = -\int dt d^3x\, \rho\, \phi$ with no average on ρ and plug in a Taylor expansion for $\phi(\mathbf{x})$ around $\mathbf{x} = \mathbf{x}_0$. Write the result as in eq. (4) and obtain explicit expressions for the multipoles $m_{ij\dots}$ in terms of $\rho(\mathbf{x})$.

II. 1PN ORBITAL EFFECTIVE ACTION

In the lecture on January 4th, it is demonstrated how to "integrate out" the orbital-scale gravito-magnetic field $A_i(\mathbf{x}, t)$ by either Fokker-action or path-integral approaches. Along these lines, let us here integrate out the gravito-electric field $\phi(\mathbf{x}, t)$ from the body-scale action

$$S^{\rm b,\phi}_{\rm eff} = \frac{1}{8\pi G}\int dt d^3x \left(\phi\Delta\phi - \frac{1}{c^2}\phi\partial_t^2\phi\right) + \sum_{a=1,2}\int dt\, m_a \left[-\phi(\mathbf{x}_a(t), t)\left(1 + \frac{3\mathbf{v}_a^2(t)}{2c^2}\right) - \frac{1}{2c^2}\phi^2(\mathbf{x}_a(t), t)\right] + \mathcal{O}(c^{-4}). \tag{5}$$

The worldlines of the two bodies are $x_a^\mu = x_a^\mu(t)$ and for the time component it holds $x_a^0(t) = ct$ (the label $a = 1, 2$ enumerates the bodies). The velocities are $\mathbf{v}_a = d\mathbf{x}_a/dt$ and $\mathbf{v}_a^2 = \mathbf{v}_a \cdot \mathbf{v}_a$. We use boldface for 3-dimensional spatial vectors, e.g. $\mathbf{x}_a = (x_a^i)$.

a) Let us familiarize ourselves with B. DeWitt's "condensed notation" introduced in the lecture, which represents integrals over x by an index contraction and in this sense is an extension of Einstein's summation convention to integrals. We assume throughout that boundary/surface terms can be dropped. We write a function of spacetime coordinates x^i, t (a scalar field) as a vector with indices $\mathbf{x}, t$, that is $f_{\mathbf{x}t} = f(\mathbf{x}, t)$ or $g_{\mathbf{x}t} = g(\mathbf{x}, t)$, and introduce the integration convention $f_{\mathbf{x}t} g_{\mathbf{x}t} = \int dt d^3x\, f(\mathbf{x}, t) g(\mathbf{x}, t)$. The position of the indices $\mathbf{x}, t$ (up or down) does not play a role here. The 3-dimensional Dirac delta function is denoted as $\delta_{\mathbf{x}\mathbf{x}'} = \delta(\mathbf{x} - \mathbf{x}') = \delta_{\mathbf{x}'\mathbf{x}}$, the one dimensional one as $\delta_{tt'} = \delta(t - t')$, the Laplacian is defined as $\Delta_{\mathbf{x}\mathbf{x}'} = \partial_i \partial_i \delta_{\mathbf{x}\mathbf{x}'}$, and its Green's function is $G_{\mathbf{x}\mathbf{x}'} = -1/(4\pi|\mathbf{x} - \mathbf{x}'|)$. Convince yourself of the following identities,

$$\delta_{\mathbf{x}\mathbf{x}'} f_{\mathbf{x}'} = f_{\mathbf{x}}, \qquad \Delta_{\mathbf{x}\mathbf{x}'} f_{\mathbf{x}'} = \partial_i \partial_i f(\mathbf{x}) = f_{\mathbf{x}'} \Delta_{\mathbf{x}'\mathbf{x}}, \qquad \Delta_{\mathbf{x}\mathbf{y}} G_{\mathbf{y}\mathbf{x}'} = \delta_{\mathbf{x}\mathbf{x}'}, \tag{6}$$

where we omitted a possible time index t on f for brevity. The last equation implies that $\Delta^{-1}_{\mathbf{x}\mathbf{x}'} = G_{\mathbf{x}\mathbf{x}'}$. Also show that

$$\Delta^{-2}_{\mathbf{x}\mathbf{x}'} := \Delta^{-1}_{\mathbf{x}\mathbf{y}} \Delta^{-1}_{\mathbf{y}\mathbf{x}'} = -\frac{1}{8\pi} |\mathbf{x} - \mathbf{x}'|. \tag{7}$$

Hint: look at $\partial_i \partial_i |\mathbf{x} - \mathbf{x}'|$.

b) Show that the action (5) can be written in condensed notation as

$$S^{\mathrm{b},\phi}_{\mathrm{eff}} = \frac{1}{2} \phi^{\mathbf{x}t} M^\phi_{\mathbf{x}t,\mathbf{x}'t'} \phi^{\mathbf{x}'t'} + J^\phi_{\mathbf{x}t} \phi^{\mathbf{x}t}, \tag{8}$$

where

$$M^\phi_{\mathbf{x}t,\mathbf{x}'t'} = M^{\phi,0}_{\mathbf{x}t,\mathbf{x}'t'} + \frac{1}{c^2} M^{\phi,1}_{\mathbf{x}t,\mathbf{x}'t'}, \tag{9}$$

$$M^{\phi,0}_{\mathbf{x}t,\mathbf{x}'t'} = \frac{1}{4\pi G} \Delta_{\mathbf{x}\mathbf{x}'} \delta_{tt'}, \tag{10}$$

$$M^{\phi,1}_{\mathbf{x}t,\mathbf{x}'t'} = \frac{1}{4\pi G} \delta_{\mathbf{x}\mathbf{x}'} \frac{\partial^2 \delta_{tt'}}{\partial t \partial t'} - \sum_{a=1,2} \int dt_a\, m_a\, \delta_a^{\mathbf{x}t}(t_a) \delta_a^{\mathbf{x}'t'}(t_a), \tag{11}$$

$$J^\phi_{\mathbf{x}t} = -\sum_{a=1,2} \int dt_a\, m_a \left(1 + \frac{3\mathbf{v}_a^2(t_a)}{2c^2}\right) \delta_a^{\mathbf{x}t}(t_a). \tag{12}$$

and we defined $\delta_a^{\mathbf{x}t}(t_a) = \delta(\mathbf{x} - \mathbf{x}_a(t_a))\, \delta(t - t_a)$.

c) Demonstrate that the inverse "matrix" of $M^\phi_{\mathbf{x}t,\mathbf{x}'t'}$, defined by $M^\phi_{\mathbf{x}t,\mathbf{y}\tau} (M^\phi)^{-1}_{\mathbf{y}\tau,\mathbf{x}'t'} = \delta_{\mathbf{x}\mathbf{x}'} \delta_{tt'}$, is approximately given by

$$(M^\phi)^{-1}_{\mathbf{x}t,\mathbf{x}'t'} = (M^{\phi,0})^{-1}_{\mathbf{x}t,\mathbf{x}'t'} - \frac{1}{c^2} (M^{\phi,0})^{-1}_{\mathbf{x}t,\mathbf{y}\tau} M^{\phi,1}_{\mathbf{y}\tau,\mathbf{y}'\tau'} (M^{\phi,0})^{-1}_{\mathbf{y}'\tau',\mathbf{x}'t'} + \mathcal{O}(c^{-4}), \tag{13}$$

to first order in $1/c^2$, and that $(M^{\phi,0})^{-1}_{\mathbf{x}t,\mathbf{x}'t'} = 4\pi G \Delta^{-1}_{\mathbf{x}\mathbf{x}'} \delta_{tt'}$. ($\tau$ is *not* the proper time here, just an integration variable.)

d) Integrate out $\phi^{\mathbf{x}t}$ using the Fokker-action approach: Obtain the equation of motion for $\phi^{\mathbf{x}t}$ from the action (8) and insert a solution for $\phi^{\mathbf{x}t}$ into the action (8). Express the result in terms of $(M^\phi)^{-1}_{\mathbf{x}t,\mathbf{x}'t'}$ and $J^\phi_{\mathbf{x}t}$ for now and denote the result as $S^{\mathrm{o},\phi}_{\mathrm{eff}} = S^{\mathrm{b},\phi}_{\mathrm{eff}}[\phi\text{-solution inserted}]$. Now, let us derive $S^{\mathrm{o},\phi}_{\mathrm{eff}}$

following an alternative approach by performing the path integral over the gravito-electric field $\phi^{\mathbf{x}t}$, leading from the body-scale (b) effective action $S^{\mathrm{b},\phi}_{\mathrm{eff}}$ to the orbit-scale (o) one $S^{\mathrm{o},\phi}_{\mathrm{eff}}$ by

$$\mathrm{const}\cdot\exp\left(\frac{i}{\hbar}S^{\mathrm{o},\phi}_{\mathrm{eff}}\right)=\int\mathcal{D}\phi\,\exp\left(\frac{i}{\hbar}S^{\mathrm{b},\phi}_{\mathrm{eff}}\right). \tag{14}$$

Calculate this Gaussian integral (treating the integral over $\phi^{\mathbf{x},t}$ as an integral over a high-dimensional vector space), and again express the result in terms of $(M^{\phi})^{-1}_{\mathbf{x}t,\mathbf{x}'t'}$ and $J^{\phi}_{\mathbf{x}t}$. Compare this result for $S^{\mathrm{o},\phi}_{\mathrm{eff}}$ to the Fokker-action approach.

e) Finally, plug in eqs. (13) and (12) into the expression that you found for $S^{\mathrm{o},\phi}_{\mathrm{eff}}$, expand the condensed notation into integrals, and simplify them. In this process, drop divergent self-interaction contributions. Assemble all pieces for the action $S^{\mathrm{o}}_{\mathrm{eff}}=S^{\mathrm{o,kin}}_{\mathrm{eff}}+S^{\mathrm{o},\phi}_{\mathrm{eff}}+S^{\mathrm{o},A}_{\mathrm{eff}}$, with $S^{\mathrm{o,kin}}_{\mathrm{eff}}$ and $S^{\mathrm{o},A}_{\mathrm{eff}}$ given in the lectures[1], and recover the 1PN action describing the orbital dynamics,

$$S^{\mathrm{o}}_{\mathrm{eff}}=\int dt\left[-m_1c^2-m_2c^2+\mathcal{L}\right], \tag{15a}$$

$$\mathcal{L}=\mathcal{L}_{\mathrm{N}}+\mathcal{L}_{\mathrm{1PN}}+\mathcal{O}(c^{-4}), \tag{15b}$$

$$\mathcal{L}_{\mathrm{N}}=\frac{m_1}{2}\mathbf{v}_1^2+\frac{m_2}{2}\mathbf{v}_2^2+\frac{Gm_1m_2}{r}, \tag{15c}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{1PN}}=&\frac{1}{8c^2}m_1\mathbf{v}_1^4+\frac{1}{8c^2}m_2\mathbf{v}_2^4-\frac{G^2m_1m_2(m_1+m_2)}{2c^2r^2}\\&+\frac{Gm_1m_2}{c^2r}\left(\frac{3}{2}\mathbf{v}_1^2+\frac{3}{2}\mathbf{v}_2^2-\frac{7}{2}\mathbf{v}_1\cdot\mathbf{v}_2-\frac{1}{2}\mathbf{v}_1\cdot\mathbf{n}\,\mathbf{v}_2\cdot\mathbf{n}\right),\end{aligned} \tag{15d}$$

where $r=|\mathbf{x}_1-\mathbf{x}_2|$, $\mathbf{n}=(\mathbf{x}_1-\mathbf{x}_2)/r$, and the dependence on the time t was suppressed here. (You might have to perform partial integrations to remove accelerations from the action.) The integrand $\mathcal{L}$ is the 2-body Lagrangian at 1PN order, also known as the Einstein-Infeld-Hoffman Lagrangian.

f) Argue why one can drop the divergent self-interaction terms, based on the fact that the point-particles (singular Dirac-delta sources) actually represent objects of finite size (cf. exercise I). **[optional!]**

III. CENTRAL-FORCE PROBLEM AT 1PN ORDER

Starting from the 1PN-Lagrangian $\mathcal{L}\approx\mathcal{L}_{\mathrm{N}}+\mathcal{L}_{\mathrm{1PN}}$ in eq. (15), in the coordinates $\mathbf{r}_1\equiv\mathbf{x}_1$, $\mathbf{r}_2\equiv\mathbf{x}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$:

a) Derive the canonical momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. [Recall from classical mechanics that $\mathbf{p}_a=\partial\mathcal{L}/\partial\mathbf{v}_a$.] Then, introduce the variables $\mathbf{R}=\mathbf{r}_1+\mathbf{r}_2$, $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, $\mathbf{P}=(\mathbf{p}_1+\mathbf{p}_2)/2$, and $\mathbf{p}=(\mathbf{p}_2-\mathbf{p}_1)/2$, and show that $\mathbf{P}$ is conserved.

b) Obtain the relative-motion Hamiltonian $H=\mathbf{p}_1\cdot\mathbf{v}_1+\mathbf{p}_2\cdot\mathbf{v}_2-\mathcal{L}$ at 1PN order in the variables $\mathbf{r}$, $\mathbf{p}$, $M=m_1+m_2$ and $\nu=m_1m_2/M^2$. [Hint: in carrying out the calculation here and below keep only terms at 1PN order! It is also strongly suggested to use Mathematica to manipulate long algebraic expressions.]

c) Compute the binding energy $E=H$ and orbital angular momentum L at 1PN order for circular orbits. Express the final result for E and L in terms of the velocity $v\equiv(M\Omega)^{1/3}$, where Ω is the

[1] These extra contributions read $S^{\mathrm{o,kin}}_{\mathrm{eff}}=\sum_{a=1,2}\int dt\,m_a[-c^2+\mathbf{v}_a^2/2+\mathbf{v}_a^4/(8c^2)]$ and $S^{\mathrm{o},A}_{\mathrm{eff}}=-4Gm_1m_2\mathbf{v}_1\cdot\mathbf{v}_2/(c^2r)$.

orbital frequency. [Hint: Impose the circular orbit condition and derive the relation between r and Ω. You will find a few new terms at 1PN order beyond the usual Newtonian relation $M/r^3 = \Omega^2$. You might find it convenient to work with Hamilton's equations in spherical coordinates and choose the motion to be in the equatorial plane.]

d) Compute the periastron advance at 1PN order for nearly circular orbits. [Hint: It is more convenient to employ the relative-motion Lagrangian. Use the conservation of energy and angular momentum to derive the equation for the radial perturbation around a circular orbit and compute the radial frequency Ω_r as function of Ω. The fractional advance of the periastron per radial period is $\Delta\Phi/(2\pi) = K(\Omega) - 1$, where $K(\Omega) = \Omega/\Omega_r$.] **[optional!]**

e) Study the stability of circular orbits using the 1PN Hamiltonian. **[optional!]**

Consider the polar coordinates (r, ϕ, p_r, p_ϕ) and a perturbation of the circular orbit defined by

$$\begin{aligned} p_r &= \delta p_r \,, \\ p_\phi &= p_\phi^0 + \delta p_\phi \,, \\ r &= r_0 + \delta r \,, \\ \Omega &= \Omega_0 + \delta\Omega \,, \end{aligned}$$

where r_0, Ω_0 and p_ϕ^0 refer to the unperturbed circular orbit. Write down the Hamilton equations and linearize them around the circular orbit solution. You should find

$$\begin{aligned} \delta\dot{p}_r &= -A_0\,\delta r - B_0\,\delta p_\phi \,, \\ \delta\dot{p}_\phi &= 0\,, \\ \delta\dot{r} &= C_0\,\delta p_r \,, \\ \delta\Omega &= B_0\,\delta r + D_0\,\delta p_\phi \,, \end{aligned} \tag{16}$$

where A_0, B_0, C_0 and D_0 depend on the unperturbed orbit. Determine explicitly A_0, B_0, C_0 and D_0.

Look at solutions of Eqs. (16) proportional to $e^{i\sigma t}$ and find the criterion of stability. [Hint: you should find that there exists a combination Σ_0 of A_0, B_0, C_0 and D_0 such that when $\Sigma_0 > 0$ the orbits are stable. The innermost stable circular orbit (ISCO) corresponds to $\Sigma_0 = 0$].

Express Σ_0 as function of $v = (M\Omega)^{1/3}$ and show that for any value of the binary mass ratio the ISCO at 1PN order coincides with the Schwarzschild ISCO. [This is an accident, which does not hold at high PN orders!]

Finally, show that $\Sigma_0 = 0$ coincides with $\Omega_r = 0$. What is the physical meaning of this result?