Instructor: Alessandra Buonanno (alessandra.buonanno@hu-berlin.de)
Guest Lecturer: Jan Steinhoff (jan.steinhoff@hu-berlin.de)
Guest Lecturer: Justin Vines (justin.vines@hu-berlin.de)
Tutor *(corresponding for this sheet)* **:** Lorenzo Speri (lorenzo.speri@hu-berlin.de)
Tutor : Stefano Savastano (stefano.savastano@aei.mpg.de)

Course webpage: https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/

Homework due date: Homeworks must be uploaded before Monday 14/12/2020 at the following address: https://moodle.hu-berlin.de/mod/assign/view.php?id=2598871

Homework rules: Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

Grading system: The homework sheet will be graded with an overall score within *0, 1, 2*.

0: not sufficient, the student has done less than half of the problems and did not attempt all of them.

1: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

2: good, the student correctly solved almost all the problems.

Recommended readings:

1. P.C. Peters & J. Mathews, Phys. Rev. **131**, 435 (1963).
2. C. Cutler et al., *The last three minutes*, Phys. Rev. Lett. **70**, 2894 (1993)

I. GRAVITATIONAL REDSHIFT OF GRAVITATIONAL WAVES

a) Let us consider a massless particle moving in a given spacetime metric $g_{\mu\nu}$. Assuming that $g_{\alpha\beta,0} = 0$ show that the component $p_0 = g_{0\mu}p^\mu$ of the particle's 4-momentum is conserved. [Hint: use the geodesic equation for the massless particle.]

b) Now, consider gravitational waves traveling through the spacetime of a non-spinning black hole. In appropriate coordinates (t, r, θ, ϕ) the spacetime metric has the Schwarzschild form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\, d\Omega^2 \,. \tag{1}$$

Let the gravitational waves have a reduced wavelength small compared to the hole's size, $\lambda \ll 2M$, and small compared to the radii of curvature of their phase fronts. Thus, geometric optics is a good approximation.

Consider a graviton moving along a ray of the waves and a family of observers who are at rest with respect to the black hole. Compute the 4-velocity of the observers at rest and show that the energy of the graviton as measured by the observers at rest is

$$E = \frac{-p_0}{\sqrt{1-2M/r}}\,. \tag{2}$$

Describe what the above formula implies as the graviton travels to larger and larger radii r.

c) As discussed in class, in geometric optics, if the waves travels precisely radially through the black-hole spacetime, then the amplitude of the wave fields decreases as $1/r$. Assume that these radially traveling waves are monochromatic and show that the gradient of their phase must have the form $\phi = \sigma(t - r_*)$ where $r_* = r + 2M\ln(r/(2M) - 1)$. [Hint: show that the gradient of this phase function is null and has $k_0 = p_0$ constant.

d) What is the energy E of a graviton for these waves, measured by the at-rest observer in terms of the constant σ? What is the frequency that the observer measures?

e) Combining the above results show that the radially traveling, monochromatic waves have the form

$$h = \frac{A}{r}\cos[\sigma\,(r_* - t) + \delta]\,, \tag{3}$$

where δ and A are constants.

II. HOW DOUBLE NEUTRON STARS FORM

In order to form a double neutron star system out of the evolution of a massive binary, the original stars must both explode as supernovae. When the second star explodes, it is probably common that more than half of the total system mass leaves suddenly. For example, if the first star has already left behind a neutron star with a mass $1.4M_\odot$ and the second star, just prior to the collapse and explosion, has a mass of $10M_\odot$, then an explosion that takes away $8.6M_\odot$ of the total $11.4M_\odot$ is needed to leave behind a 1.4–$1.4M_\odot$ binary.

a) Assuming that i) the binary was in a circular orbit prior to the second supernova, ii) the supernova exploded isotropically in its rest frame and delivered no kick to the neutron star left behind, iii) the mass of the second star is lost *instantaneously* in the supernovae explosion, does the binary survive to this process?

b) How does the above result change if the supernova delivers a kick to the neutron star?

III. NEUTRON-STAR BINARY SYSTEMS

Consider a binary system composed of two $1.4M_\odot$ neutron stars in a circular orbit.

a) Starting from the time evolution equation for the orbital frequency computed in HW1,

$$\dot{\omega}^3 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} G^5 \mu^3 M^2\,, \tag{4}$$

show that the rate of change of the orbital period is

$$\frac{dP}{dt} = -\frac{192\pi}{5}\,\frac{m_1\,m_2}{(m_1+m_2)^2}\left(\frac{2\pi G(m_1+m_2)}{c^3\,P}\right)^{5/3} \tag{5}$$

where m_1 and m_2 are the neutron star masses, and compute dP/dt in $\mu\,\mathrm{s\,yr}^{-1}$ when $P = 7.75\mathrm{h}$.

b) If the orbital period is 7.75 h, how long will it be until the neutron stars collide?

c) How far apart can the companions be if the time to collision is less than 10^{10} yr?

d) How much time remains before collision once the gravitational-wave emission frequency reaches 40 Hz? 100 Hz?

e) Assume the neutron star binary is at a distance $r = 1\,\mathrm{Mpc}$ and inclination $\theta = 0$, i.e. face-on, plot (e.g., using Mathematica) the polarization $h_+(t)$ and the frequency $f(t)$ versus time for gravitational-wave frequencies between 100 and 300 Hz.

IV. MASS OCTUPOLE RADIATION FROM A BINARY SYSTEM

a) Let us consider a binary system of masses m_1 and m_2 and reduced mass μ whose center-of-mass coordinate moves along the circular orbit

$$x(t) = R\cos\omega\, t \qquad y(t) = R\cos\iota\sin\omega\, t \qquad z(t) = R\sin\iota\sin\omega\, t\,,$$

where $R^2 = x^2 + y^2 + z^2$. Let us set the observer's direction $\hat{n}$ along the z direction. [Henceforth, the (x, y, z) axes are also labeled as $(1, 2, 3)$.] The octupole (oct) gravitational radiation is

$$(h_{ij}^{\mathrm{TT}})_{\mathrm{oct}} = \frac{1}{r}\frac{2G}{3c^5}\,\Lambda_{ij,kl}(\hat{n})\,\dddot{M}_{kl3}\,,$$

where r is the distance of the observer from the binary and

$$M^{klm} = \mu\,\frac{\delta m}{m}\,x^k\,x^l\,x^m \qquad \Lambda_{ij,kl} = P_{ik}\,P_{jl} - P_{ij}\,P_{kl}/2 \qquad P_{ij} = \delta_{ij} - \hat{n}_i\,\hat{n}_j$$

with $\delta m = m_1 - m_2$ and $m = m_1 + m_2$. Show that $(h_+)_{\mathrm{oct}}$ and $(h_\times)_{\mathrm{oct}}$ components are

$$(h_+)_{\mathrm{oct}} = \frac{1}{r}\frac{G\mu R^3\omega^3}{12c^5}\frac{\delta m}{m}\sin\iota\left[\left(3\cos^2\iota - 1\right)\cos\omega t - 27\left(1 + \cos^2\iota\right)\cos 3\omega t\right]\,, \tag{6}$$

$$(h_\times)_{\mathrm{oct}} = \frac{1}{r}\frac{G\mu R^3\omega^3}{12c^5}\frac{\delta m}{m}\sin 2\iota\left[\sin\omega t - 27\sin 3\omega t\right]\,. \tag{7}$$

At which frequencies the octupole gravitational radiation is emitted?

b) *Current-quadrupole and power radiated*

With an analogous calculation, one can show the contribute to GWs from the current quadrupole (cq) is

$$\left(h_{ij}^{TT}\right)_{\mathrm{cq}} = \frac{1}{r}\frac{4G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})n_m(\epsilon^{mkp}\ddot{J}^{p,l} + \epsilon^{mlp}\ddot{J}^{p,lk})\,, \qquad J^{i,j}(t) = \mu\frac{\delta m}{m}\epsilon^{ikl}x^k(t)\dot{x}^l(t)x^j(t)\,. \tag{8}$$

The components $(h_+)_{\mathrm{cq}}$ and $(h_\times)_{\mathrm{cq}}$ for the binary described in IVa read as

$$(h_+)_{\mathrm{cq}} = \frac{1}{r}\frac{4G\mu R^3\omega^3}{3c^5}\frac{\delta m}{m}\sin\iota\cos\omega t\,, \tag{9}$$

$$(h_\times)_{\mathrm{cq}} = \frac{1}{r}\frac{2G\mu R^3\omega^3}{3c^5}\frac{\delta m}{m}\sin 2\iota\sin\omega t\,. \tag{10}$$

Determine the power radiated from the quadrupole and the octupole using the following formula,

$$P_{\mathrm{oct+cq}} = \frac{r^2\,c^3}{16\pi G}\,2\pi\int_{-1}^{1} d\cos\iota < \dot{h}_+^2 + \dot{h}_\times^2 > .s$$

Express the ratios $P_{\mathrm{oct+cq}}(\omega)/P_{\mathrm{quad}}(2\omega)$ and $P_{\mathrm{oct+cq}}(3\omega)/P_{\mathrm{quad}}(2\omega)$, where $P_{\mathrm{quad}} = 32G\mu^2R^4\omega^6/(5c^5)$, in terms of v/c using the Newtonian relation between R and ω, and estimate those ratios for $v/c = 10^{-2}$.