**Instructor:** Alessandra Buonanno (`alessandra.buonanno@hu-berlin.de`)
**Guest Lecturer:** Jan Steinhoff (`jan.steinhoff@hu-berlin.de`)
**Guest Lecturer:** Justin Vines (`justin.vines@hu-berlin.de`)
**Tutor** *(corresponding for this sheet)* **:** Lorenzo Speri (`lorenzo.speri@hu-berlin.de`)
**Tutor :** Stefano Savastano (`stefano.savastano@aei.mpg.de`)

**Course webpage:** `https://imprs-gw-lectures.aei.mpg.de/2020-gravitational-waves/`

**Homework due date:** Homeworks must be uploaded before Monday 30/11/2020 at the following address: `https://moodle.hu-berlin.de/course/view.php?id=99604`

**Homework rules:** Homeworks must be neat, and must either be typed or written in pen (not pencil!). Please do not turn in homework that is messy or that has anything that's been erased and written over (or written over without erasing), making it harder to read.

**Grading system:** The homework sheet will be graded with an overall score within *0, 1, 2*.

*0*: not sufficient, the student has done less than half of the problems and did not attempt all of them.

*1*: sufficient, the student has done more than half of the problems and she/he tried to solve almost all of them.

*2*: good, the student correctly solved almost all the problems.

**Recommended readings:**

1. Local Lorentz and free-falling frames, Sec. 8.4 in J. Hartle, "Gravity".
2. Proper detector frame, see, e.g., W. Ni and M. Zimmermann, Phys. Rev. D **17**, 1473 (1978).
3. Newtonian and relativistic tidal gravity, see e.g., Sec. 24.2-24.5 in R.D. Blandford and K.S. Thorne, http://www.pma.caltech.edu/Courses/ph136/ph136.html
4. Gravitational lensing, see: J. Wambsganss, Living Rev. Relativ. (1998) 1: 12 https://dx.doi.org/10.12942/lrr-1998-12

**Exercises:**

1. **Resonant mass detectors**

   The first gravitational-wave detector was a resonant mass detector or bar detector. It was built at the University of Maryland by Joseph Weber in the late 60's. It was a large, heavy, metal bar. The bar would absorb an impinging gravitational wave and be set into oscillations. Hopefully, those oscillations would be detectable.

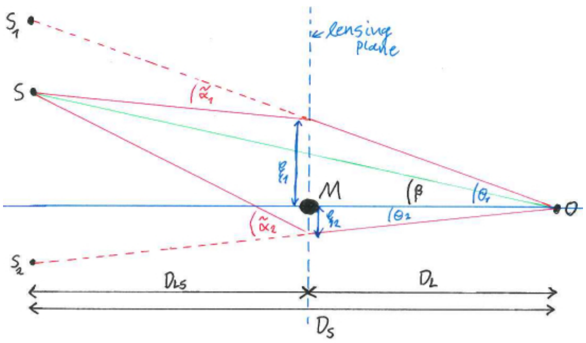
   The simplest way to model a bar detector is with a damped spring. Let us assume that we have two masses $m_1$ and $m_2$, along the $x$-axis, connected by a spring with spring constant $k$ and subjected to a dissipative force $F_{\rm diss} = -b\, dx/dt$, where $x(t)$ is the displacement of the masses. In equilibrium the masses are separated by a length $L$. Let us assume that a plane gravitational wave with frequency $\omega$ arrives along the $z$-axis and it is polarized only along the $x$-axis, i.e., $h_+ = h \cos\omega\, t, h_\times = 0$.

- Assuming that $L \ll \lambda_{\rm GW}$, with $\lambda = 2\pi c/\omega$, show that the equation of motion for the displacement of the masses $x(t)$ with respect to the equilibrium position are of the form:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 \frac{1}{2} h L \cos(\omega t) - \gamma \frac{\mathrm{d}x}{\mathrm{d}t} - \omega_0^2 x + \mathcal{O}(h^2) \tag{1}$$

  where $\gamma$ and $\omega_0^2$ are constants dependent on $b, m_1, m_2, k$. Find how $\gamma$ and $\omega_0^2$ are related to $b, m_1, m_2, k$. [Hint: follow the description in the local inertial frame.]
- The solution of the equation of motion in the previous item can be written as $x(t) = A\,\cos(\omega\, t + \delta)$. Derive $A$ and $\delta$ and find the resonance frequency $\omega_r$, i.e. the frequency that maximizes the amplitude $A$. Determine also $A(\omega_r)$ and $\delta(\omega_r)$. Express your results in terms of $h, L, \gamma, \omega_0^2$.
- Derive the kinetic energy of the oscillations, the potential energy of the oscillator, the work done on the oscillator by the gravitational wave and the rate of energy dissipated. Compute those quantities averaging over a cycle of oscillations.
- Assume that $h = 10^{-21}$, $L = 1\,\mathrm{m}$, reduced mass $\mu = m_1\, m_2/(m_1 + m_2) = 1000\,\mathrm{kg}$, quality factor $Q = 10^6$ and $f = \omega_0/(2\pi) = 1\,\mathrm{kHz}$, compute the maximum of oscillations at resonance and the total energy in the oscillations, i.e., kinetic energy and potential energy, averaging over a cycle of oscillations. Remember that the quality factor is $Q = 2\pi \frac{\text{energy stored}}{\text{energy dissipated per cycle}}$.
  When comparing the averaged total energy to the thermal energy at room temperature, what do you conclude about the possibility of measuring gravitational waves with such a system?

2. **Gravitational lensing of gravitational waves**




Consider a gravitational wave burst (a short-duration gravitational-wave signal) in the geometric optics limit. As it travels from the source $S$ to the observer $O$ on Earth, the gravitational-wave rays (red) are deflected, or "lensed," by a heavy (point-)mass $M$. In addition, during this trip to the observer, the gravitational potential of the point mass affects the time of arrival to the observer. For solving this exercise, assume that the rays and the mass $M$ are in the same plane, and that the distances $D_L$, $D_{LS}$ are large compared to $\xi_1$, $\xi_2$, i.e., the deflection is confined to a "lensing plane", and that the deflection angles are $\tilde{\alpha}_i = 4M/\xi_i$.

Calculate:

- the time delay due to only the lensing effect, i.e. the time it takes for the signal from $S$ to $O$ along the two red paths in the above figure. Use the fact that the considered angles are small and notice that $\tilde{\alpha}_1 \approx D_S(\theta_1 - \beta)/D_{LS}$ and $\tilde{\alpha}_2 \approx D_S(\theta_2 + \beta)/D_{LS}$.
- the time delay due to only the gravitational potential of the point mass $M$, which is called Shapiro delay. The approximate line element reads

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right) dx^i dx^i + O\left(\frac{M^2}{r^2}\right), \tag{2}$$

  where $r^2 = x^i x^i$. You can also assume that the path from $S$ to $O$ is a straight line at a distance $\xi_i$ from the point mass.

What is the difference in arrival times between the two rays (red paths) for a burst like GW150914 (distance $D_S = 1$ Gly) passing a galaxy of $10^{12}$ solar masses in the middle of its way and at an angle of $\beta = 2$ arsec? Compare this to the burst duration of $\sim$ 200 ms observed in the detector. What are the magnification factors of the two rays? Remember that the magnification factors are given by $\mu_i = (1 - (\theta_E/\theta_i)^4)^{-1}$, where $\theta_E = \sqrt{4MD_{LS}/(D_L D_S)}$.

3. **Energy-momentum tensor of a plane gravitational wave**

Calculate the effective energy-momentum tensor of a plane gravitational wave in TT gauge from

$$T^{\mathrm{GW}}_{\mu\nu} = \frac{1}{32\pi} < \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} >, \tag{3}$$

$$h^{\mu\nu} = \left[(e^\mu_x e^\nu_x - e^\mu_y e^\nu_y)A_+ + (e^\mu_x e^\nu_y + e^\mu_y e^\nu_x)A_\times\right] e^{ik_\alpha x^\alpha} + c.c. \tag{4}$$

It holds $e^\mu_x e_{y\mu} = e^\mu_x k_\mu = e^\mu_y k_\mu = k^\mu k_\mu = 0$, $e^\mu_x e_{x\mu} = e^\mu_y e_{y\mu} = 1$, and $A_+, A_\times \in \mathbb{C}$.

4. **Attenuation of gravitational waves**

Assume that a gravitational wave encounters a viscous fluid, which is initially at rest with fluid four-velocity given by $u^\mu = (1, 0, 0, 0)$.

- The shearing of the fluid is described by the shear tensor

$$\sigma_{\mu\nu} = \frac{1}{2}\nabla_\mu u_\nu + \frac{1}{2}\nabla_\nu u_\mu + \frac{1}{2}u_\mu u^\alpha \nabla_\alpha u_\nu + \frac{1}{2}u_\nu u^\alpha \nabla_\alpha u_\mu - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)\nabla_\alpha u^\alpha \,. \tag{5}$$

  Show that the shear has purely spatial components when the gravitational-wave is expressed in the transverse-traceless (TT) gauge, and that

$$\sigma_{ij} = \frac{1}{2}\frac{\partial}{\partial t} h^{\mathrm{TT}}_{ij} \,. \tag{6}$$

- The shearing of the viscous fluid generates a contribution to the stress-energy tensor of the form

$$T_{\mu\nu} = -2\eta\, \sigma_{\mu\nu} \,, \tag{7}$$

  where we indicate with $\eta$ the coefficient of viscosity of the fluid. What is the linearized field equation for the gravitational wave in the TT gauge in presence of the viscous fluid? Show that a plane wave travelling along the $z$-axis is attentuated by the fluid by an amount $e^{-z/l}$ where $l$ is the attenuation length scale $l = c^3/(8\pi G\eta)$.
- Chocolate has a coefficient of viscosity of $\eta = 25\,\mathrm{kg}/(\mathrm{m\,s})$. Calculate the distance $L$ that a gravitational wave must travel through chocolate before it is attentuated by a factor $1/e$.